2022

Auckland Mathematical Olympiad

Problems & Solutions

1. Each of the 10 dwarfs either always tells the truth or always lies. It is known that each of them loves exactly one type of ice cream: vanilla, chocolate or fruit. First, Snow White asked those who like the vanilla ice cream to raise their hands, and everyone raised their hands, then those who like chocolate ice cream - and half of the dwarves raised their hands, then those who like the fruit ice cream - and only one dwarf raised his hand. How many of the gnomes are truthful?

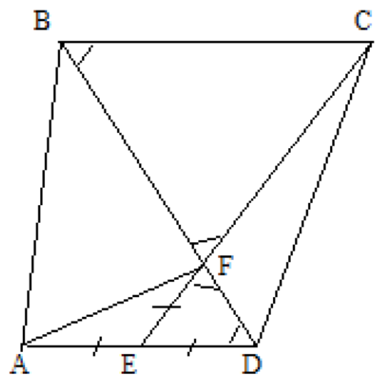
   Solution. The dwarves who always tell the truth raised their hand once, and the dwarves who always lie raised their hand twice. A total of $16 = 10 + 5 + 1$ hands were raised. If all the dwarves told the truth, 10 hands would be raised. If one truthful gnome is replaced by one liar, then the number of hands raised will increase by 1. Since six "extra" hands were raised, six gnomes lied, and four told the truth.

   Answer: 4. □

2. The number 12 is written on the whiteboard. Each minute, the number on the board is either multiplied or divided by one of the numbers 2 or 3 (a division is possible only if the result is an integer) . Prove that the number that will be written on the board in exactly one hour will not be equal to 54.

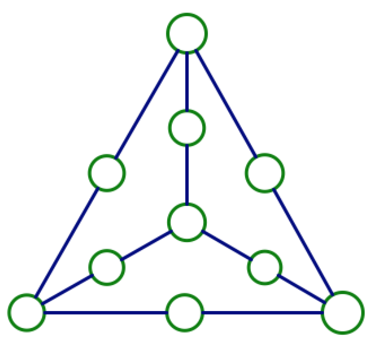
   Solution. After each division and each multiplication, the parity of the total number of 2's and 3's in the prime factorisation of the number on the board changes. At the beginning, the number of primes is 3 which is odd: $12 = 2 \times 2 \times 3$. Therefore, after 60 operations it should be odd, but in the factorisation of the number $54 = 2 \times 3 \times 3 \times 3$ the number of primes is 4 which is even. □

3. Point E is the midpoint of the base AD of the trapezoid $ABCD$. Segments BD and CE intersect at point F. It is known that AF is perpendicular to BD. Prove that $BC = FC$.



Solution. By the property of the median of a right triangle $EF = ED$, therefore, the angle $\angle EDF$ is equal to the angle $\angle EFD$. Note that angles $\angle EDF$ and $\angle FBC$ are equal as internal crosses lying at parallel lines AD and BC, angles $\angle EFD$ and $\angle BFC$ are equal as vertical. Thus, in triangle $\triangle BFC$, angles $\angle B$ and $\angle F$ are equal, so $BC = CF$. □

4. Is it possible to arrange all the integers from 0 to 9 in circles so that the sum of three numbers along any of the six segments is the same?



Solution. Let's assume it's possible. Let the sum of the four numbers at the ends of the segments be equal to A, the sum of the six numbers located in the middle of the segments is equal to B, and the sum of the three numbers along each segment is equal to C. Then $A + B = 0 + 1 + \cdots + 9 = 45$. Each end point belongs to exactly three segments, and all midpoints are distinct. Therefore, adding up the sums along all six segments, we get $3A+B=6C$. Hence $2A=6C-(A+B)=6C-45$. But this is impossible, since $2A$ is an even number, and $6C-45$ is an odd number. The resulting contradiction proves that the required arrangement is impossible. $\square$

5. The teacher wrote on the board the quadratic polyomial $x^2+10x+20$. Then in turn, each of the students came to the board and increased or decreased by 1 either the coefficient at x or the constant term, but not both at once. As a result, the quadratic polyomial $x^2+20x+10$ appeared on the board. Is it true that at some point a quadratic polyomial with integer roots appeared on the board?

   Solution. Note that with each change in the coefficients of the polynomial, its value at the point $x=-1$ changes by 1 (in one direction or another). The value of the first quadratic polynomial

$$f(x)=x^2+10x+20$$

   at this point is equal to $f(-1)=11$, and the value of the last,

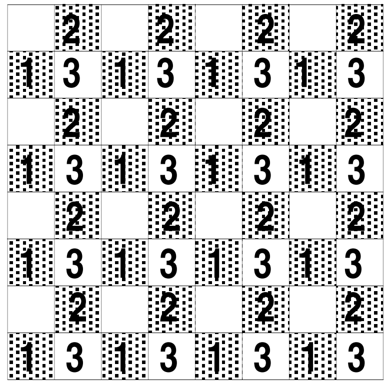
$$g(x)=x^2+20x+10,$$

   is $-g(-1)=-9$. Therefore, at some intermediate moment, a polynomial

$$h(x)=x^2+px+q$$

   was written on the board for which $h(-1)=0$. Both of its roots are integers: one is equal to -1, the other, according to the Vieta theorem, is equal to $-q$ which is also an integer. $\square$

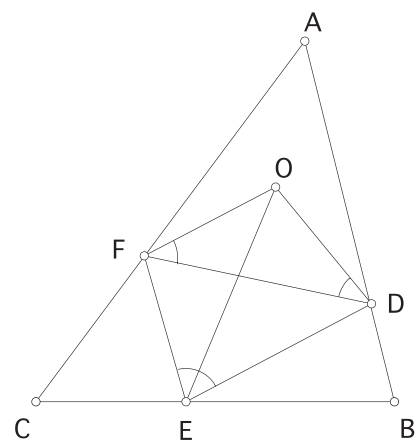
6. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there are an even number of pieces on the black squares of the board.



Solution. Let us number columns from left to right and rows from bottom to top. We refer to the black squares of odd numbered rows as to the squares of the first kind, and the black squares of even numbered rows will be called the squares of the second kind. Finally, we say that all white squares of rows with odd numbers are squares of the third kind. Suppose that there are n_i pieces on the squares of the ith kind. From the conditions of the problem we have that $n_1 + n_3 = 4$ and $n_2 + n_3 = 4$. Therefore $n_1 = n_2$, and the number of pieces on black squares is equal to $n_1 + n_2 = 2n_1$ which is an even number. □

7. Points D, E, F are chosen on the sides AB, BC, AC of a triangle ABC, so that $DE = BE$ and $FE = CE$. Prove that the centre of the circle circumscribed around triangle ADF lies on the bisectrix of angle DEF.

   Solution. Since



$$\angle DEF = 180^\circ - (180^\circ - 2\angle B) - (180^\circ - 2\angle C) = 180^\circ - 2\angle A,$$

then $\angle A$ is acute. Consequently, the centre O of the circle circumscribed around triangle ADF and the vertex A lie on the same side of segment FD, and therefore,

$$\angle DOF = 2\angle A = 180^\circ - \angle DEF.$$

It follows that the points O, D, E, F lie on the same circle, and hence

$$\angle DEO = \angle DFO = \angle FDO = \angle FEO,$$

so that EO is the bisectrix of $\angle DEF$. □

8. Find the least value of the expression $(x+y)(y+z)$, under the condition that x, y, z are positive numbers satisfying the equation

$$xyz(x + y + z) = 1.$$

Solution. For arbitrary positive x, y, z satisfying the equation $xyz(x + y + z) = 1$, we have

$$(x + y)(y + z) = (x + y + z)y + xz = \frac{1}{xz} + xz \geq 2.$$

This inequality becomes an equality, for example, when $x = z = 1$ and $y = \sqrt{2} - 1$. □

9. Does there exist a function $f(n)$, which maps the set of natural numbers into itself and such that for each natural number $n > 1$ the following equation is satisfied

$$f(n) = f(f(n-1)) + f(f(n+1))?$$

Answer: There are no such functions.

Solution. If such a function existed, then among all its values

$$f(2), f(3) \ldots, f(n), \ldots$$

there would be a minimal one, say $f(n_0)$, $n_0 > 1$. Note that

$$f(n_0+1) \geq f(n_0) = f(f(n_0-1)) + f(f(n_0+1)) \geq 1+1 > 1,$$

Due to minimality of $f(n_0)$, it is true that $f(f(n_0+1)) \geq f(n_0)$, which implies that

$$f(n_0) = f(f(n_0-1)) + f(f(n_0+1)) \geq 1 + f(n_0),$$

which is not possible. □

10. It is known that $\frac{7}{13} + \sin\phi = \cos\phi$ for some real ϕ. What is $\sin 2\phi$?

Answer: $\frac{120}{169}$.

Solution. By squaring the equation $\frac{7}{13} = \cos\phi - \sin\phi$ we get

$$\frac{49}{169} = \cos^2\phi + \sin^2\phi - 2\sin\phi\cos\phi = 1 - \sin 2\phi.$$

So $\sin 2\phi = 1 - \frac{49}{169} = \frac{120}{169}$. □

11. For which k the number $N = 101\cdots0101$ with k ones is a prime?

Answer: There is only one such prime: 101 so $k = 2$.

To see this, suppose that $N = 101\cdots0101$ with k ones, for some $k \geq 2$. Then

$$99N = 9999\cdots9999 = 10^{2k} - 1 = (10^k+1)(10^k-1).$$

If moreover N is prime, then N divides either $10^k + 1$ or $10^k - 1$, and hence one of $\frac{99}{10^k-1} = \frac{10^k+1}{N}$ and $\frac{99}{10^k+1} = \frac{10^k-1}{N}$ is an integer. For $k > 2$, $10^k - 1$ and $10^k + 1$ are both greater than 99, so we get a contradiction. Therefore $k = 2$ and $N = 101$ (which is prime).

12. There are 11 empty boxes. In one move, a player can put one coin in each of some 10 boxes. Two people play, taking turns. The winner is the player after whose move in one of the boxes there will be 21 coins. Who has a winning strategy?

    Solution. We number the boxes: $1, 2, \ldots, 11$ and denote the move by the number of the box where we did not put the coin. We can assume that the first player started the game with move 1. To win, the second player must, regardless of the moves of the first, make moves $2, 3, \ldots, 11$. With these ten moves, together with the move of the first, 10 coins will be placed in each box by the second player.

    In addition, there is box (let's call it A) into which the first player in moves from 2d to 11th put in a coin. Thus, after the 11th move of the first one in the box A there will be 20 coins, and no box will contain more. The second player on his 11th move must put coins in such a way that a coin gets into box A. Thus, he wins. $\square$