Auckland Mathematical Olympiad

Problems

1. A single section at a stadium can hold either 7 adults or 11 children. When N sections are completely filled, an equal number of adults and children will be seated in them. What is the least possible value of N?

   Answer: The least common multiple of 7 and 11 is 77. Therefore, there must be 77 adults and 77 children. The total number of sections is $\frac{77}{7} + \frac{77}{11} = 11 + 7 = 18$.

2. Triangle ABC of area 1 is given. Point A' lies on the extension of side BC beyond point C with $BC = CA'$. Point B' lies on extension of side CA beyond A and $CA = AB'$. C' lies on extension of AB beyond B with $AB = BC'$. Find the area of triangle $A'B'C'$.

   Solution. Let us prove that each of the triangles $A'B'C$, $A'BC'$, $A'B'C'$ has area 2 so that the area of large triangle is 7. Let us look at $AB'C'$ and draw the line BB'. We have

   $$S_{ABC} = S_{ABB'} = S_{BB'C'}$$

   so $S_{AB'C'} = 2$. □

3. Each square on an 8×8 checkers board contains either one or zero checkers. The number of checkers in each row is a multiple of 3, the number of checkers in each column is a multiple of 5.

   Assuming the top left corner of the board is shown below, how many checkers are used in total?

   Solution. **Answer:** 30 checkers. The total number of checkers T, must be a multiple of 3 (adding all rows). It must also be a multiple of 5 (adding all columns). 3 and 5 are both prime, hence the only numbers which are multiples of both must be multiples of 15, hence $T = 15k$ for some k. The depicted corner of the board has 2 checkers, hence the total number of checkers is > 0. Every column has a multiple of 5, but also has a maximum of 8, hence the maximum possible number is 5 per column, hence $T \leq 5 \times 8 = 40$. Hence, $T = 15$ or $T = 30$. Currently we have two checkers in different columns. Each of these columns must have 5 checkers. Assuming we have 15 checkers in total, this indicates that we have only one remaining non-zero column but both the first and second row need to have *two* additional checkers in them to reach a multiple of three, leading to (at least) four non-zero columns, ie $T \geq 4 \times 5 = 20$. The only multiple of 15 in the appropriate range is 30, therefore 30 checkers have been used. □

4. Which digit must be substituted instead of the star so that the following large number is divisible by 7?

$$\underbrace{66\cdots66}_{2023}\star\underbrace{55\cdots55}_{2023}$$

Solution. Since 111111 is divisible by 7 we need to find the digit that makes $6\star5$ divisible by 7. Such digit is 6. □

5. There are 11 quadratic equations on the board, where each coefficient is replaced by a star. Initially, each of them looks like this

$$\star x^2 + \star x + \star = 0.$$

Two players are playing a game making alternating moves. In one move each of them replaces one star with a real nonzero number.

The first player tries to make as many equations as possible without roots and the second player tries to make the number of equations without roots as small as possible.

What is the maximal number of equations without roots that the first player can achieve if the second player plays to her best? Describe the strategies of both players.

Solution. **Answer:** 6 equations.

Let us make the following observation: if the first player makes the coefficient of x to be 1, i.e., makes one of the equations $\star x^2+x+\star = 0$, then, whatever the second player puts there, the first can immediately finish it with roots present. For example, if the second player puts a instead of any of the two remaining stars, the first puts $1/a$ instead of the remaining stars achieving the determinant $D = 1 - 4a\cdot\frac{1}{a} = -3 < 0$.

The strategy of the first player is as follows: he tries to put 1 as coefficient of x in as many untouched equations as possible. However, if the second player responded with putting a in one of those, he finishes that equation off as described above before putting 1 instead of x in one of the untouched equations. This way he can put 1 as coefficient of x in 6 equations. After that in the next several moves he puts arbitrary numbers in the remaining 5 equations. We note that this is the second player who will eventually start putting numbers in the first 6 equations.

This way the first player can convert 6 undetermined equations with coefficients with roots. He cannot do more if the second player sticks with the following tactics. He must put 1 as coefficient of x^2 in as many untouched equations as possible. If the first payer puts a as coefficient of x, the second responds with -1 as the constant term. If the first player puts c as the constant term, the second player responds with putting $b > 2\sqrt{|c|}$ as coefficient of x. □

6. Suppose there is an infinite sequence of lights numbered $1, 2, 3, \ldots$, and you know the following two rules about how the lights work:

   - If the light numbered k is on, the lights numbered $2k$ and $2k+1$ are also guaranteed to be on.
   - If the light numbered k is off, then the lights numbered $4k+1$ and $4k+3$ are also guaranteed to be off.

   Suppose you notice that light number 2023 is on. Identify all the lights that are guaranteed to be on?

   Solution. We note that at least one light is on. Let k be a number with a light on. We argue that if $k > 1$ then a light corresponding to a smaller number is on, too.

   We can write k in one of the forms: $4m+2, 4m+3, 4m+4, 4m+5$ (where $m \geq 0$). If $k = 4m+3$ then the light numbered m is on. If $k = 4m+5 = 4(m+1)+1$, then the light numbered $m+1$ is on. If $k = 4m+2$, then the light numbered $2(4m+2)+1 = 8m+3$ is on, but $8m+3 = 4(2m)+3$ so the light numbered $2m$ is on. Finally, if $k = 4m+4$, then the light numbered $2(4m+4)+1 = 8m+9$ is on, but $8m+9 = 4(2m+2)+1$ so the light numbered $2m+2$ is on. In each case we have shown a light with a smaller number is on.

   From this argument it follows that the light numbered 1 is on.

   Now we claim all the lights are on. Let $k > 0$ be the first number corresponding to a light which is not on; we can write either $k = 2m+1$ or $k = 2m$. In either case $m < k$ and the light numbered m is on by assumption. Therefore so is the light numbered k, giving a contradiction. □

7. In a square of area 1 there are situated 2024 polygons whose total area is greater than 2023. Prove that they have a point in common.

   Solution. Let's denote the figures by $F_1, \ldots, F_{2024}$ and by $G_1, \ldots, G_{2024}$ denote their complements to the square. As the total area of $F_1, \ldots, F_{2024}$ is greater than 2023, then the total area of their complements $G_1, \ldots, G_{2024}$ is less than 1 and they do not cover the square. But this is the same to say that $F_1, \ldots, F_{2024}$ have a point in common. □

8. How few numbers is it possible to cross out from the sequence

   $$1, 2, 3, \ldots, 2023$$

   so that among those left no number is the product of any two (distinct) other numbers?

Solution. It is clear that, if we remove 43 numbers $2, 3, \ldots, 44$, then, since $45^2 = 2025$, among those left no one is the product of any two others. This is the minimal number. To prove that consider 43 triples $(k, 89-k, (89-k)k)$, for $k = 2, \ldots, 44$. They do not have numbers in common and we have to remove at least one number from every such triple. □

9. Quadrillateral $ABCD$ is inscribed in a circle with centre O. Diagonals AC and BD are perpendicular. Prove that the distance from the centre O to AD is half the length of BC.

Solution. Draw the chord AE perpendicular to DA. Its length is twice the distance from O to AD and it is sufficient to prove that it is equal to BC. Let us do some calculation of arcs. As diagonals are perpendicular, $\smile CB = 180° - \smile AD$. But as DE is a diameter, $\smile EA = 180° - \smile AD$. The two arcs are equal, and hence the chords. □

10. Find the maximum of the expression

$$||...||x_1 - x_2| - x_3| - \ldots | - x_{2023}|,$$

where $x_1, x_2, \ldots, x_{2023}$ are distinct natural numbers between 1 and 2023.

Solution. **Answer:** 2022.

Since for $x \geq 0$, $y \geq 0$ the inequality $|x - y| \leq \max\{x, y\}$ holds, then by induction

$$|| \ldots ||x_1 - x_2| - x_3| - \ldots | - x_n| \leq \max\{x_1, x_2, \ldots, x_n\}$$

and hence the maximum of the expression in question is not greater than 2023. This value 2023 can not however be achieved since

$$||...||x_1 - x_2| - x_3| - \ldots | - x_{2023}| \equiv x_1 + x_2 + \ldots + x_{2023} = 1012 \cdot 2023 \pmod 2,$$

and, in particularly, even.

2022, however, can be achieved as the following example shows.

$$\begin{aligned} &||||| \ldots |||| \ldots |||||1 - 2| \\ &\quad -3| - 5| - 6| - 4| - \ldots \\ &\cdots - (4k-1)| - (4k+1)| - (4k+2)| - 4k| - \ldots \\ &\cdots - 2020| - 2021| - 2022| - 2020| - 2023| \\ &\qquad = |1 - 2023| = 2022. \end{aligned}$$

It is easy to understand it if to have in mind that $|1 - 2| = 1$ and $|1 - (4k-1)| - (4k+1)| - (4k+2)| - 4k| = 1$ for any $k \geq 1$. □

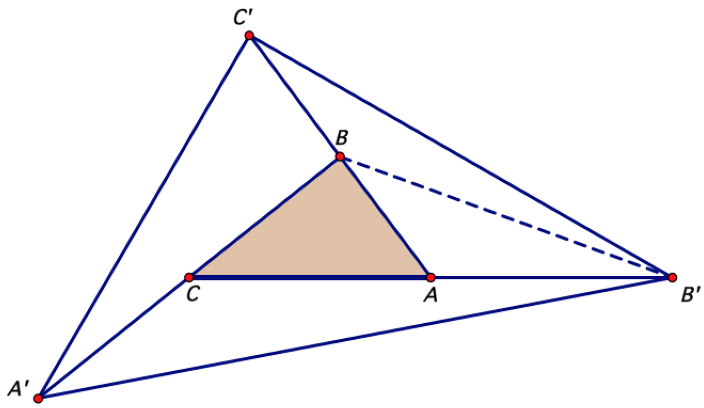

Figure 1: Junior geometry picture.