Department of Mathematics

Abstracts of the Student Research Conference 2020

(Total 25 abstracts)

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Infinitely many multi-hump solitons in a waveguide with quartic dispersion.

Ravindra Bandara
Supervisors : Bernd Krauskopf and Neil Broderick

Optical solitons are signals that do no change their shape as they are transmitted along a waveguide. Of particular interests are pure quartic solitons, which have been observed experimentally in [1] and exist through the balance between the quartic dispersion and the Kerr nonlinearity of the (photonic crystal) waveguide. Unlike conventional solitons, pure quartic solitons do not have quadratic dispersion. A natural question is whether new optical solitons with both quadratic and quartic dispersion exist. Most recently, Tam et al [2] tackled this problem by showing theoretically that such solitons do exist; namely, they used the generalized nonlinear Schrödinger equation (GNLSE) with fourth-order dispersion as a model to describe the transmission of a signal along the photonic crystal waveguide.

We consider a stationary wave solution ansatz, where the optical pulse does not undergo a change in shape while propagating. This allows us to transform the governing partial differential equation (PDE) into a fourth-order nonlinear ordinary differential equation (ODE). We study the ODE to find bi-asymptotic trajectories, known as homoclinic solutions, which correspond to soliton solutions of the PDE. We take advantage of the mathematical properties of reversibility and existence of a Hamiltonian of the ODE to show that there are infinitely many symmetric homoclinic solutions in a certain parameter regime. Specifically, we employ numerical continuation techniques to compute a representative number of them. Moreover, we show the existence of infinitely many non-symmetric soliton solutions that arise in the ODE as connecting orbits between periodic orbits and equilibrium solutions.

References:
1. Blanco-Redondo, A., De Sterke, C. M., Sipe, J. E., Krauss, T. F., Eggleton, B. J., & Husko, C. (2016). Pure-quartic solitons. *Nature Communications*, *7*, 10427

2. Kevin K. K. Tam, Tristram J. Alexander, Andrea Blanco-Redondo, and C Martijn De Sterke. Generalized dispersion kerr solitons. Physical Review A, 101, 2020.

Dissipative chaos in a Kerr coupled ring resonator of counter-propagating light with spontaneous symmetry-breaking

Rodrigues Bitha

Micro-ring resonators represent the next generation of chip-scale optical systems offering low power and high performance. Mathematically, the propagation of light in these devices is modelled by dissipative wave equations, with damping, nonlinear and coupling terms, that exhibit different sets of symmetries specific to the device geometry. In particular, we study the dynamics of counter-propagating light in a micro-ring resonator [1], where laser light is split in equal power and sent into a microresonator in opposite directions. This scenario is described by a four-dimensional system of ordinary differential equations, where the intensity and the detuning of the light pumped into the device are parameters. In this talk, we show the existence of localised high/low intensity waves and chaotic attractors in the vicinity of period-doubling cascades emanating from Hopf bifurcations. The period-doubling cascades accumulate, in parameter space, on the so-called wild Shilnikov homoclinic bifurcations. In particular, we continue this homoclinic orbit in parameter space and find that it is organized by codimension-two points, namely, Belyakov transitions and points of heteroclinic cycles [2]. Finally, we show how the dynamics of the ring resonator are organized in different regions of parameter space near these codimension-two points. Overall, our work showcases the distinct and exotic behaviour that should be expected from this optical device.

Keywords: Symmetry, Counter-propagative light, Schilnikov bifurcation, Belyakov transition, heteroclinic cycles.

<u>Collaborator:</u> Andrus Giraldo
<u>Supervisors:</u> Neil Broderick and Bernd Krauskopf.

References:

[1] Micheal Woodley *et al., "Universal symmetry-breaking dynamics for the Kerr interaction of counterpropagating light in dielectric ring resonator*", Phys. Rev. A **98**, 053863 (2020).

[2] Mikhail Shashkov *et al.*, " *On the complex bifurcation set for a system with simple dynamics*", Int. Jour. Bif. and Chaos **6**, 5 (1996).

DIMENSION OF GENERALIZED INVERSE LIMITS

Álvaro Menéndez Calzada
Supervisor: Sina Greenwood

August 2020

Abstract

Given a sequence $\{X_i\}_{i\in\mathbf{N}}$ of Hausdorff compact spaces and a sequence of upper-semicontinuous set-valued functions $\mathbf{f} = \left\{f : X_{i+1} \to 2^{X_i} \mid i \in \mathbb{N}\right\}$, the generalized inverse limit (GIL) of $\mathbf{f}$ is defined as the set of points $\mathbf{x}$ in the product $\Pi_{i\in\mathbb{N}} X_i$ such that each $x_i \in f_{i+1}(x_{i+1})$ (where $x_i = \pi_i(\mathbf{x})$). GILs extend the classical notion of an inverse limit for which the functions are simply continuous functions, and $\mathbf{x}$ is contained in the inverse limit if $x_i = f_{i+1}(x_{i+1})$. The dimension of an inverse limit on 1-dimensional spaces can be at most one, whereas a GIL on 1-dimensional spaces can have any dimension, even infinite.

For the case when the X_i are intervals, it has been proved in [1] that the GIL has dimension at least n if and only if $\mathbf{f}$ admits a weighted sequence $\langle \tau_1, \cdots \tau_{2n-2}\rangle$. One objective of my research is to generalize that result to more complex sppaces. Currently I am working on the version of the mentioned result for circles and finite graphs.

References

[1] I. Banič and S. Greenwood. A characterisation of the dimension of inverse limits of set-valued functions on intervals. *Fundamenta Matematicae*, 249, 2020.

[2] W. T. Ingram. *An Introduction to Inverse Limits with Set-Valued Functions.* Springer, New York, 2012.

Mathematical billiards in quadrilaterals

Henry Chen
Supervisor: Professor Hinke Osinga

Abstract

Mathematical billiards is an area of study in dynamical systems, where we model billiards in an idealised environment. We stipulate that the billiard ball is a point mass that satisfies the law of reflection when interacting with the boundary. These simple constraints lead to surprisingly deep and complex dynamics. The shape of the mathematical billiard is arbitrary and we focus on quadrilateral boundaries. Polygons with interior angles of the form $\frac{p}{q}\pi$, $\frac{p}{q} \in \mathbb{Z}$, are well understood and can be studied using a process called unfolding [1]. For the simplest cases, unfolding transforms the billiard trajectory to become a straight line within the tessellation of the plane. We present the necessary and sufficient conditions in order for a trajectory to be periodic in the square. Then we extend the results to the rectangular billiard and demonstrate how unfolding fails for general parallelogram billiards. We show several unexpected and interesting applications for mathematical billiards including the pouring problem/riddle [2].

References

[1] Tabachnikov, S. (2005). "Geometry and billiards". American Mathematical Society Student Mathematical Library Vol. 30.

[2] Polster, B., and Ross, M. (2012). "Math Goes to the Movies". JHU Press.

Hurwitz Numbers and the Semi-infinite Wedge

Student: Xavier Coulter
Supervisor: Pedram Hekmati

The study of Hurwitz numbers includes a vast array of areas of mathematics, from combinatorics to topology and complex analysis. Hurwitz numbers originate from considering isomorphism classes of holomorphic maps between smooth complex 1-dimensional manifolds. We quickly see the local structure of these maps is restrictive enough to give a correspondence between isomorphism classes of these maps and ramified coverings of real surfaces. The main contribution to this correspondence is given by Riemann's existence theorem [2, p.159][1, p.84].

The transition of Hurwitz numbers from the study of complex analytic structures to that of ramified coverings of surfaces naturally leads itself to the idea of monodromy representation i.e. group homomorphisms from the fundamental group of the target surface $\pi_1(X)$ to the symmetric group S_d, specified by structure of the ramified covering. It turns out, that any homomorphism from $\pi_1(X)$ to S_d contains enough information to construct a ramified covering admitting the given homomorphism as its monodromy representation.

This opens further avenues into representation theory of the symmetric group. The key result of this approach is given by Burnside's character formula, which gives a closed formula for any given Hurwitz number as a sum of products of irreducible characters [1, p.124]. This concludes the largely classical survey of results on Hurwitz numbers.

The focus of this presentation will be on a relatively recent result, concerning the relation between Hurwitz numbers, and the semi-infinite wedge space $\Lambda^{\frac{\infty}{2}}V$. To give some context, $\Lambda^{\frac{\infty}{2}}V$ can very naturally be interpreted as a description of Dirac's electron sea in many-particle physics – hence the name semi-infinite, as all negative states past some certain point must be filled. We take V to be a complex vector space with basis indexed by half integers $\{\dots, v_{-3/2}, v_{-1/2}, v_{1/2}, v_{3/2}, \dots\}$. Then $\Lambda^{\frac{\infty}{2}}V$ is a vector space with basis

$$v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \dots$$

where from some i_N we have $i_n = i_N - (N - n)$ for all $n \geq N$ – this is the semi-infinite condition. Certain Hurwitz numbers then arise as expectation values of operators on $\Lambda^{\frac{\infty}{2}}V$ [3]. In particular, operators are defined whose action on the basis elements of $\Lambda^{\frac{\infty}{2}}V$ is reminiscent of the Murnaghan-Nakayama rule, used for calculating irreducible characters of the symmetric group. Burnside's character formula then immediately bridges the gap for certain examples of Hurwitz numbers. A natural question to ask then is how one can generalise this operator-Hurwitz number correspondence, for a broader range of Hurwitz numbers.

References

[1] Renzo Cavalieri and Eric Miles. *Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory.* London Mathematical Society Student Texts. Cambridge University Press, 2016.

[2] M. Fried. *Riemann's Existence Theorem.* Cambridge Studies in Advanced Mathematics. Cambridge University Press.

[3] Paul Johnson. *Double Hurwitz Numbers via the Infinite Wedge.* Transactions of the American Mathematical Society. 2010.

Accumulators and Hidden Order Groups in Cryptography

Samuel Dobson

Supervised by Steven Galbraith

Suppose we have a large, secret set S, and want to be able to prove to others that some $x \in S$. We want to do this in a compact way, a short proof - without, say, revealing the entire set. We also want assurance that we can't make fake proofs of membership for non-elements $y \notin S$.

Why might we want to do this? One interesting use-case is in cryptocurrencies like Bitcoin [MGGR13]. We can store the entire set of "spendable" coins S in one compact representation, and prove that we are allowed to spend a coin by proving that it is contained in the set. This gives a compact representation of the entire blockchain state, at the same time as providing privacy, because we don't know where the coin came from - just that it is inside the set.

This construction is called a cryptographic **Accumulator** [BdM94]. In this presentation, we will show some known ways of creating such an accumulator, each with its own tradeoffs. Specifically, we are interested in accumulators created using **Hidden Order Groups** (groups whose order is unknown to everyone). We will give a brief overview of our recent work in this area and some of the results and improvements we have made [DG20].

References

[BdM94] Josh Benaloh and Michael de Mare. One-way accumulators: A decentralized alternative to digital signatures. In *Advances in Cryptology — EUROCRYPT '93*, pages 274–285. Springer Berlin Heidelberg, 1994.

[DG20] Samuel Dobson and Steven D. Galbraith. Trustless groups of unknown order with hyperelliptic curves. Cryptology ePrint Archive, Report 2020/196, 2020. `https://eprint.iacr.org/2020/196`.

[MGGR13] Ian Miers, Christina Garman, Matthew Green, and Aviel D Rubin. Zerocoin: Anonymous distributed e-cash from bitcoin. In *2013 IEEE Symposium on Security and Privacy*, pages 397–411. IEEE, 2013.

Hypersurfaces and the Second Fundamental Form in Projective Differential Geometry

Simon Goodwin
Supervisor: Rod Gover

The study of hypersurfaces and submanifolds is of great interest in differential geometry, the theory in the classical Riemannian case is well understood, and in the conformal case, many results exist including with applications to conformal compactication.

Motivated by projective compactification and the strong theory in other parabolic geometries, we have been investigating hypersurfaces in projective and metric projective differential geometry. Including behaviour of the familiar second fundamental form.

In this talk I will give a brief introduction to projective structures, give a result relating to Codazzi tensors and relate this to the study of hypersurfaces and the second fundamental form.

Analytic Torsion and the Cheeger-Müller Theorem

Elizabeth Jagersma
Supervised by Pedram Hekmati

In algebraic topology, torsion is a generalization of the determinant from a single operator to a collection of operators called a cochain complex. In 1935 K. Reidemeister computed the torsion for operators called combinatorial Laplacians which act on a cochain complex built out of topological spaces. Remarkably, this produces a quantity that is independent of these operators and only depends on the underlying topological space. This torsion invariant has extremely useful applications.

When the topological space has a smooth structure, there also exists analytic Laplacians which act on infinite dimensional vector spaces. Recall that the determinant is the product of eigenvalues. For analytic Laplacians the determinant will be a diverging product of eigenvalues and hence ill-defined. However, it is possible to force this infinite product to become finite by suppressing the larger eigenvalues at an exponential rate. This is achieved via the method of zeta function regularization.

We call the torsion computed from the combinatorial Laplacians the Reidemeister torsion, and the torsion computed from the analytic Laplacians the Ray-Singer torsion, after D. Ray and I. Singer, who originally computed the analytic torsion [RS71]. The Cheeger-Müller theorem says that the Reidemeister torsion and Ray-Singer torsion are equal. This theorem was proven independently in the celebrated papers of J. Cheeger [Che79] and W. Müller [Mül78]. Both proofs were topological in nature.

In 1992, J. M. Bismut and W. Zhang discovered a new proof of the Cheeger-Müller theorem which also vastly generalises the result [BZ92]. Their proof is analytic in nature and introduces a deformation of the analytic Laplacian by a smooth function called a Morse function. The resulting operator is known to have a gap in its set of eigenvalues which factorizes the Ray-Singer torsion into a small and large component. By computing the asymptotics of these two components separately, it is possible to directly show that the Ray-Singer and Reidemeister torsions coincide.

While extremely elegant, the 1992 Bismut-Zhang proof involved some very difficult analysis which has since been simplified. In 1994, Bismut and Zhang simplified the analysis of the small component of the torsion [BZ94], and in 2003 M. Braverman circumvented the difficult analysis of the large torsion by introducing a clever comparison analysis of the large component of the Ray-Singer torsion [Bra03].

This talk will outline the beautiful ideas underlying torsion and the proof of the Cheeger-Müller theorem in broad strokes.

References

[Bra03] M Braverman. "New proof of the Cheeger-Müller theorem". In: *Annals of global analysis and geometry.* 23.1 (2003), pp. 77–92. ISSN: 0232-704X.

[BZ92] J.-M. Bismut and W. Zhang. "An extension of a theorem by Cheeger and Müller". en. In: Astérisque 205 (1992), pp. 7–218.

Hamiltonian circuits in circulant directed graphs

Anna Inae Jeong
Supervisor: Gabriel Verret

Let $n \in \mathbb{N}$ and let $X \subseteq \mathbb{Z}_n$. The *circulant directed graph* $\mathrm{C}(n, X)$ is the directed graph with vertex set $\mathbb{Z}_n$ and arc set $\{(u, v) | v - u \in X\}$.

A *directed Hamiltonian circuit* is a directed circuit such that each and every vertex of G appears exactly once.

It is of interest to determine whether there exists a Hamiltonian circuit in circulant directed graphs. Until present, the Hamiltonicity of circulant directed graphs which have two outgoing arcs at each vertex have been studied completely by Rankin (1948). I would like to demonstrate the proof of this theorem which I have reproduced.

Unlike circulant directed graphs which have two outgoing arcs at each vertex, less is known about circulant directed graphs with three or more outgoing arcs (at each vertex). Based on the results for these circulant directed graphs, I have used a computer programme to construct circulant directed graphs and check for their Hamiltonicity. I would like to share some of these examples which I find interesting.

Reference

Rankin, R. A. (1948, January). A campanological problem in group theory. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 44, No. 1, pp. 17-25). Cambridge University Press.

A heteroclinic cycle as a controller in an evolutionary robotics task

Valerie Jeong

Supervisors: Claire Postlethwaite and Matthew Egbert

In the field of evolutionary robotics, a control system of a robot is optimised using a genetic algorithm that is analogous to natural evolution. The controller can be described as a continuous dynamical structure, and we are particularly interested in the use of a heteroclinic cycle as a controller in evolutionary robotics. This idea is motivated by [2], suggesting that heteroclinic cycles can explain the "chunking" phenomenon observed when the brain processes information.

In realistic settings, controllers are not free from perturbations and noise. However, the combined effect of perturbations and noise to a heteroclinic cycle is not well understood. In particular, it was observed that the residence time near a steady state increases for sufficiently small noise level, which is an unexpected phenomenon [1].

This talk is split into two parts. First, we present a plausible analytic explanation for the unexpected phenomenon with a stochastic model for a noisy trajectory near a heteroclinic cycle. In the second part, we present the use of a heteroclinic cycle in a restricted problem of a hexapod locomotion task, considering only a single leg. We evolve a set of parameters of a two-node heteroclinic cycle and demonstrate that the evolved robot shows a nearly periodic solution.

References

[1] P. Ashwin and C. M. Postlethwaite, "Sensitive finite-state computations using a distributed network with a noisy network attractor," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, pp. 5847–5858, 2018.

[2] M. Rabinovich, P. Varona, I. Tristan, and V. Afraimovich, "Chunking dynamics: Heteroclinics in mind," *Frontiers in Computational Neuroscience*, vol. 8, p. 22, 2014.

Wild chaos, blenders and heterodimensional connections.

Dana Contreras Julio*
Supervisors: Hinke Osinga, Bernd Krauskopf

Wild chaos is the name given to a new type of chaotic dynamics that can arise only in systems of sufficiently high dimension. To construct an abstract example of higher-dimensional dynamical systems that exhibit wild chaos, Bonatti and Díaz introduced the notion of a blender in 1996 [1]. The characterising feature of a blender is that its invariant manifolds behave as geometric objects of a dimension that is larger than expected from the dimensions of the manifold itself. This feature may imply the existence of complicated and robust heterodimensional connections between different parts of phase space, which can lead to wild chaos.

The first explicit example and image of a blender was introduced by S. Hittmeyer *et al.* in 2018 [2]. It is a family of three-dimensional Hénon-like diffeomorphisms that have two saddle fixed points. Unfortunately, both fixed points have the same index and there is no heterodimensionality in the system. In this talk, we construct a new map based on this example that exhibits a heterodimensional connection and investigate the generation and disappearance of blenders. We illustrate why heterodimensional connections and blenders are closely related with wild chaos. Moreover, we show preliminary images of candidate blenders, corresponding to one-dimensional manifolds of fixed points that are computed with advanced numerical techniques. We also discuss how to check numerically whether those manifolds correspond to blenders.

References

[1] C. Bonatti and L.J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms, *Annals of Mathematics* (2), **143** (1996), 357–396.

[2] S. Hittmeyer, B. Krauskopf, H.M. Osinga and K. Shinohara, Existence of blenders in a Hénon-like family: geometric insights from invariant manifold computations, *Nonlinearity* (10), **31** (2018), R239–R267.

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Simulating nectarivorous avian foraging behavior and pollen dispersal in New Zealand

M. Riley Knoedler
advised by Claire Postlethwaite

August 2020

Abstract

Native New Zealand plants that rely on bird visitation for successful pollination exhibit evidence of reproductive failure and inbreeding depression. It is an open question whether alien plant species overall negatively impact native plant neighbors by pulling birds away from them (on a local scale), or whether aliens can provide a net benefit to natives by attracting more birds to the area overall (on a regional scale). Gathering sufficient empirical data of pollen dispersal in the context of alien plants can be prohibitively time consuming. Additionally, the coupling of more detailed and biologically realistic, but computationally expensive, agent-based models with large scale dynamical systems models is an important technique in ecological modelling. In this talk, I will describe a spatially explicit, stochastic agent-based model of the nectar foraging behavior of competing bird species under several proposed avian memory paradigms. I will discuss my investigation of pollination outcomes, in particular the dispersal kernel, which is the probability distribution function describing the distance pollen is dispersed from its parent plant, in a variety of landscape compositions and configurations. I will also examine challenges that emerge in using demographic parameters derived from the output of the agent-based model to parameterize the classic Lotka-Volterra competition model of population distribution.

Name: Jo Knox

Main supervisor: Igor' Kontorovich

Co-supervisors: Caroline Yoon, Fiona Ell

How can we tell if learners are seeing the general in the particular?: Discursive markers of generic example-use.

A 'generic example' (Mason & Pimm, 1984) or 'generic proof' is where a particular example is signalled within an argument as being representative of a whole class of objects, rather than the particular case itself. Generic proofs are considered to be particularly appropriate in the primary school context (e.g. Stylianides, 2007): (i) in contrast to empirical arguments, they are mathematically valid, and (ii) they have explanatory potential and offer accessibility for young learners over formal proofs. However, criteria for judging whether the author of an example-based argument is seeing the *general* and understands why it holds for all cases, or is seeing the *particular* in the example used, has been missing (Reid & Vallejo Vargas, 2018).

As part of my PhD research, I examined primary school students' verbal responses and their accompanying actions for indicators that suggested examples were being used generically rather than empirically. I analysed videos and transcripts from eights groups of 8- and 9-year-old students working in groups of four as they substantiated their endorsement of narratives about the sums of odds and evens. Using Sfard's (2008) commognitive framework, I developed four categories of example-use according to whether the example revealed numeric or generic realizations of odd and even and whether the substantiations were inductive or deductive. My findings show nuanced and multi-layered differences in example-use rather than a simple dichotomy of empiric or generic use of examples.

In this talk, I present these categories along with a range of subtle discursive markers that I claim implicitly point towards students' reasoning generically rather than empirically, and which provide criteria for researchers and educators in determining whether an example is being used generically.

References:

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277-289.

Reid, D., & Vallejo Vargas, E. (2018). When Is a Generic Argument a Proof?. In *Advances in mathematics education research on proof and proving* (pp. 239-251). Springer, Cham.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge

Stylianides, A. J. (2007). The notion of proof in the context of elementary school mathematics. Educational Studies in Mathematics, 65, 1–20.

Patterns in sets of positive density

Harris Leung
Supervisor: Jeroen Schillewaert

7 September 2020

In number theory, a natural way to measure the 'density' of a subset $A \subseteq \mathbb{N}$ is by assigning to it a limiting proportion of the natural numbers which it contains, namely $\limsup_n |A \cap \{1, \ldots, n\}|/n$. One can interpret this as the probability of encountering an element of the subset when choosing from $\{1, \ldots, n\}$ with uniform probability, as n grows.

The following is a more general analogue of this notion. Let (X, d) be a metric space endowed with a measure μ. Fix $o \in X$ and let $S_n = \{x \in X \mid d(o, x) = n\}$. The *density* of a subset $A \subseteq X$ is defined to be

$$d^*(A) = \limsup_{n\to\infty} \frac{\mu(A \cap S_n)}{\mu(S_n)}.$$

A result in geometric Ramsey theory of Furstenberg, Katznelson and Weiss [1] states that if A is a Lebesgue measurable subset of $X = \mathbb{R}^2$ with positive density then the set $\{d(x, y) \mid x, y \in A\}$ of distances between points in A contains all sufficiently large real numbers. Bourgain [2] then provided a vast generalisation of this result, showing that if a Lebesgue measurable subset of $\mathbb{R}^n$ has positive density then it contains a sufficiently large dilation of the vertex set of any $(n-1)$-simplex in $\mathbb{R}^n$.

We explore analogues of these results in the 'p-adic context' of affine buildings proved in a recent paper of Björklund, Fish and Parkinson [3]. Rank one buildings are trees, and we provide sketch proofs of the following analogue results of Furstenberg-Katznelson-Weiss and Bourgain.

Proposition. *Let X be the regular tree of degree $q+1 \geq 3$ and let $A \subseteq X$ have positive density. There exists $K > 0$ such that for all $t \in \mathbb{N}$ with $t \geq K$ there exist vertices $x, y \in A$ with $d(x, y) = 2t$ and $d(o, x) = d(o, y)$.*

Theorem. *Let X be the regular tree of degree $q+1 \geq 3$ and let $A \subseteq X$ have positive density. For each $k > 0$ there exists $K > 0$ such that whenever $t_1, \ldots, t_k \in \mathbb{N}$ with $t_k \geq \cdots \geq t_2 \geq t_1 \geq K$ there exists a subset $\{v_0, v_1, \ldots, v_k\} \subseteq A$ with $d(v_0, v_j) = 2t_j$ for all $1 \leq j \leq k$ and $d(o, v_0) = d(o, v_1) = \cdots = d(o, v_k)$.*

We end with a discussion of the result proved in this paper for higher rank buildings and mention some further analogues and generalisations in progress.

References

[1] H. Furstenberg, Y. Katznelson B. Weiss, *Ergodic theory in sets of positive density.* J. Mathematics of Ramsey Theory. Algorithms and Combinatorics, vol. 5, Springer, Berlin, Heidelberg, 1989.

[2] J. Bourgain, *A Szemerédi type theorem for sets of positive density in* $\mathbb{R}^k$, Isr. J. of Math. 54, 307–316, 1986.

[3] M. Björklund, A. Fish, J. Parkinson, *Patterns in sets of positive density in trees and affine buildings*, `arXiv:0706.1234 [math.CO]`, 2019.

Maximal Peak-Pit Domains on Four Alternatives

Guanhao Li
Supervisor: Arkadii Slinko

Condorcet domains are collections of preferences such that the majority voting on every pair of alternatives yields no cycles. Peak-pit domains are those that satisfy a complete set of either never-top or never-bottom conditions. There are three well-known classical peak-pit domains: single-peaked domains, see Black (1948, 1958); alternating scheme domains, see Fishburn (1997); and single-crossing domains, see Puppe and Slinko (2016). Those three domains are connected and have maximal width.

In this talk, we classify maximal peak-pit domains on four alternatives, by confirming theoretically the existing computational classification of all Condorcet domains on four alternatives by Dittrich (2016). In particular, we show that for any maximal Condorcet domain on four alternatives, it is a peak-pit domain if and only if it is connected. We also show that over four alternatives, the graph of any maximal peak-pit domain of non-maximal width contains a four-cycle.

References

Black, D. (1948). On the rationale of group decision-making. *Journal of Political Economy*, 56:23–24.

Black, D. (1958). *The Theory of Committees and Elections*. Cambridge University Press, Cambridge.

Dittrich, T. (2016). All condorcet domains on $n \leq 4$ alternatives. *Technical report, Karlsruhe Institute of Technology.*

Fishburn, P. C. (1997). Acyclic sets of linear orders. *Social Choice and Welfare*, 14(1):113–124.

Puppe, C. and Slinko, A. (2016). Condorcet domains, median graphs and the single-crossing property. *Economic Theory*, pages 1–34.

COSET GRAPH LABELING

GEORGINA LIVERSIDGE
(SUPERVISOR: MARSTON CONDER)

A group presentation $G = \langle\, X \,|R\,\rangle$ defines a group in terms of a set of generators X and a set of relations R. A presentation $G = \langle\, X \,|R\,\rangle$ is called finite if both X and R are finite sets.

Group presentations are a simple and natural way to define groups, however they can be difficult to deal with. Simple problems, like "determine the size of the group", can be difficult to solve.

A Schreier coset graph is a graphical representation of the natural representation of the group on the cosets of a subgroup. A Schreier coset graph can be labeled to get a presentation of the given subgroup. With a careful choice of labels, it can also be used to find words for elements of the subgroup in terms of a chosen set of generators.

In this talk I will describe my implementation of coset graph labeling and explain some of its uses in analysing finitely presented groups.

What are we doing?

Gray Manicom

Supervisors: Claire Postlethwaite and Vivien Kirk

I take a break from thesis-related topics to ask the question; what are we doing? “We are doing mathematics to explain things”, would be a fairly straightforward reply, but what does “doing mathematics” mean, and how does mathematics explain anything? Watch me ponder these questions, live, at our upcoming seminar.

Name: Aidan Mason-Mackay
Supervisor: Marie Graff

Microwave Imaging Using Adaptive Eigenspace Inversion

An inverse problem is any problem where measurements from some physical process are used to deduce the physical structure that caused them. A well-known example of an inverse problem is X-ray imaging, where measurements of attenuated X-rays are used to deduce structures in the human body. For this project, we used microwave imaging to detect diseased material in tree trunks by building images of the trees interior using a method called Adaptive Eigenspace Inversion (AEI).

Input data was generated by simulating an experiment where microwaves propagate towards a tree trunk and the reflected signals are measured. A cost function was then formulated to quantify the error between the predicted and observed electric field measurements for a given guess at the physical properties of the tree trunk. This cost function was then minimised using numerical optimisation methods.

Due to having access to only partial noisy data, most real-world inverse problems are ill-posed, meaning they have no solution; infinite solutions; and/or unstable solutions. Ill-posedness is a problem because it can cause convergence to a local minimum, and it's generally addressed using standard regularisation techniques. The novelty of this project was the use of a recently developed regularisation technique, Adaptive Eigenspace Inversion, where the relative permittivity inside the tree trunk was projected onto the basis of an elliptical operator at each step of the optimisation algorithm. The advantage of the AEI method is its significant reduction of computation time and its potential for improved accuracy compared to standard methods.

The AEI method was tested on the simulated data and produced reconstructions within 3% accuracy of the true profile for the tree trunk. Furthermore, the number of electric field measurements taken around the perimeter of the tree trunk could be reduced significantly from 72 to 12 with very little loss of reconstruction accuracy. This is potentially a useful result for industry applications, as recording less electric field data could reduce the cost and time of experiments.

Ideal Hierarchical Access Structures and Lattice Path Matroids

Songbao Mo
Supervisor: Arkadii Slinko

Secret sharing schemes, first introduced by Shamir [4] (1979) and also independently by Blakley [1] (1979), are now widely used in many cryptographic protocols as a tool for securely storing information that is highly sensitive and important. Such information includes decryption keys, missile launch codes, and numbered bank accounts.

A *secret sharing scheme* is a method to distribute shares of a secret value among a set of participants. Only the qualified subsets of participants can recover the secret value from their shares. The family of all qualified subsets forms the *access structure* of the schemes. The scheme is *perfect* if the unqualified subsets can learn nothing about the secret value whatsoever. A perfect secret share scheme is *ideal* if the length of every share is the same as the length of the secret. Ideal secret sharing schemes are the most informationally efficient which is important in applications. The central problems of the theory of secret sharing schemes is to classify all access structures that can carry an ideal secret sharing scheme. A milestone paper in secret sharing by Brickell and Davenport [2] (1991) shows that an ideal access structure is always a matroid port (the concept that extensively studied in combinatorics by Seymour[3]). However, the problem appears to be very difficult and successfully solved only in some subclasses of schemes.

In this talk, I will introduce a class of well-studied access structures named hierarchical access structures and a special class of matroids named lattice path matroids. I will show that hierarchical access structures can be characterised as matroid ports of lattice path matroids. Moreover, we show that a hierarchical access structure is conjunctive or disjunctive if and only if it is a matroid port of a shifted matroid.

References

[1] G.R Blakley, Safeguarding cryptographic keys, *AFIPS Conference Proceedings* **48** (1979) 313-317.

[2] E. Brickell and D. Davenport, On the classification of ideal secret sharing schemes, *J. Cryptology* **4** (1991) 123-134.

[3] P.D. Seymour, A forbidden minor characterization of matroid ports, *Quart. J. Math. Oxford* **27** (1976) 407–413.

[4] A. Shamir, How to share a secret, *Commun. of the ACM* **22** (1979) 612-613.

[BZ94] J -M Bismut and W Zhang. "Milnor and ray-singer metrics on the equivariant determinant of a flat vector bundle". In: *Geometric and functional analysis : GAFA*. 4.2 (1994), pp. 136–212.

[Che79] J. Cheeger. "Analytic torsion and the heat equation". In: *Annals of Mathematics*. 2nd ser. 109.2 (1979), pp. 259–321. DOI: 10.2307/1971113.

[Mül78] W. Müller. "Analytic torsion and R-torsion of Riemannian manifolds". In: *Advances in Mathematics* 28 (3 1978), pp. 233–305. DOI: https://doi.org/10.1016/0001-8708(78)90116-0.

[RS71] D. B. Ray and I. M. Singer. "R-Torsion and the Laplacian on Riemannian Manifolds". In: *Advances in Mathematics* 7 (2 1971), pp. 145–210. DOI: https://doi.org/10.1016/0001-8708(71)90045-4.

A transverse index theorem for toric contact manifolds

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Abstract

One of the most significant mathematical results of the 20th century, the Atiyah-Singer index theorem connects three different branches of mathematics, algebraic topology, differential geometry, and analysis. The theorem expresses the index of linear elliptic differential operators in terms of certain topological invariants. In fact, it subsumes many deep theorems including the Gauss-Bonnet theorem, the Riemann-Roch theorem, and Hirzebruch's signature theorem. The problem of finding such an expression was originally posed in a 1960 paper by Israel Gel'fand, and it was solved by Michael Atiyah and Isadore Singer in 1962. In 1968 Atiyah and Singer presented a much more fundamental proof of the index theorem, relying on K-theory. This proof allows one to extend the index theorem in many directions, in particular it allows one to consider G-equivariant operators for a compact Lie group G and families of elliptic operators.

When considering a G-equivariant elliptic operator P the index is no longer an integer but an element of $R(G)$, the representation ring of G. This means that computing the G-index corresponds to determining the multiplicities of representations of G in the spaces ker P and coker P. Atiyah and Bott found a localization formula for the G-index similar to the classic Lefschetz fixed point formula, which evaluates the G-index by localizing to fixed point set.

The G-index theory can be extended vastly by considering operators that are not elliptic but transversally elliptic. This means that the symbol of the operator P can be inverted only in directions transverse to the G-orbits. In this case the spaces ker P and coker P are no longer finite dimensional, but the multiplicities of the representations appearing are all finite. This implies that the G-index still defines a distribution on G.

The transverse index is very difficult to compute in general, despite the existence of cohomological formulas by Berline-Vergne-Paradan. The reason being that these formulas express the G-index only in a neighbourhood of the identity element in G.

In this talk we will discuss a class of geometries which carry natural transversally elliptic operators and have enough symmetries to allow for localisation of the symbol to closed G-orbits and yield explicit index formulas.

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Development of an Assessment Self-Efficacy (ASE) Scale

Bandura (1997) defines self-efficacy as "the beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3). Self-efficacy is a significant affective construct in education due to its predictive relationship with achievement. In educational contexts, experiences of success positively influence the development of self-efficacy while experiences of failure impair it (Usher & Pajares, 2009). In particular, success or failure in one assessment may influence a student’s beliefs about future assessments. Since most postsecondary institutions measure students’ progress through summative assessment, understanding student assessment-related self-efficacy is important for educators.

Existing assessment-related self-efficacy measurements tend to concentrate on students’ beliefs about their abilities to perform specific tasks but omit features specific to assessment-taking. The aim of this research was to develop the Assessment Self-Efficacy (ASE) Scale, to more accurately account for student academic self-efficacy in assessment. This study outlines the development and validation of the scale in two postsecondary mathematics assessment conditions: an online quiz and a final exam. A confirmatory factor analysis of the 10 items in study 1 (N=301) suggested a two-factor model with latent factors emotional regulation (3 items) and comprehension and performance abilities (5 items), with the rephrasing of one item. Study 2 (N=356) of students in a stage two service mathematics course confirmed this model offered the best general fit for both assessments. The potential uses of the ASE scale for research in education are discussed.

References

Bandura, A. (1997). *Self-efficacy: The exercise of control.* New York: W. H. Freeman.

Usher, E. L., & Pajares, F. (2009). Sources of self-efficacy in mathematics: A validation study. *Contemporary Educational Psychology, 34*, 89-101. doi:doi:10.1016/j.cedpsych.2008.09.002

GENERALISING EINSTEIN

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Abstract. Einstein manifolds are a class of smooth manifolds together with metric data, which obey the Einstein equation. This condition is an equality relating the metric to a curvature object–itself determined uniquely by the metric–on the manifold. When the dimension n of the Einstein manifold is taken to be 4 (corresponding to 3 spatial dimensions and 1 time dimension) and the metric in question is of Lorentzian signature, such a manifold is precisely the object known as a spacetime in Einstein's theory of relativity.

However, interest in Einstein manifolds has long extended beyond the scope of theoretical physics, and they have captured the fascination of pure mathematicians who have found them to be rich geometric objects. Constructing explicit examples of these spaces is no easy task, however. The Einstein equation is in reality a system of coupled, nonlinear, second order partial differential equations in the coefficients of the metric in some local coordinate system, presenting an incredibly challenging problem in trying to satisfy this set of equations simultaneously.

To thicken the plot, this classical theory of Einstein manifolds has seen generalisations built onto it over the years. One way to generalise is to loosen restrictions on the geometric data which produces the curvature-object we consider in the Einstein equation. The particular loosening we have studied, which corresponds to allowing connections on the manifold which have non-vanishing and totally skew-symmetric torsion, is equivalent to equipping the manifold with an additional 3-form and modifying the Einstein equations to include the coefficients of the 3-form in a local coordinate system. These modified Einstein equations, which we shall call the generalised Einstein equations, pile another level of complexity on top of the already unwieldy original Einstein equations.

In this talk, we present the ingenious work of two mathematicians who, by picking a certain class of manifolds which is circumscribed by cherry-picked geometric data, used their wiles and trickery to construct a new solution to the analytically challenging Einstein equations. We will then discuss our attempt to modify this trick to work in the generalised Einstein setting.

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Generalised Inverse Limits

- Connectedness on Hausdorff Continua -

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Generalised inverse limits can give rise to immense complexity, despite being constructed from simple bonding functions. They have been a fruitful area of study in recent times with applications in continua theory. Given a sequence of topological spaces X_i and functions $f_i : X_{i+1} \to 2^{X_i}$ we define the generalised inverse limit by

$$\varprojlim(X_i, f_i) = \left\{ (x_0, x_1, ...) \in \prod_{i \in \mathbb{N}} X_i : \forall i \in \mathbb{N}, x_i \in f_{i+1}(x_{i+1}) \right\}$$

For traditional inverse limits on continua, the inverse limit is always connected. However, there exists generalised inverse limits from upper semicontinuous surjective functions with connected graphs that are disconnected.

Given a sequence of compact Hausdorff spaces X_i and upper semicontinuous set-valued functions $f_i : X_{i+1} \to 2^{X_i}$. Find necessary and sufficient conditions such that the inverse limit is connected.

By a result from Nall [1], a generalised inverse limit is connected if and only if every finite approximation of the inverse limit is also connected. From this a new tool was introduced to simplify results about connectedness; the Mahavier product defined by

$$\bigstar_{[1,n]}\Gamma(f_i) = \left\{ (x_0, ..., x_n) \in \prod_{i \le n} X_i : \forall i \le n, (x_i, x_{i-1}) \in \Gamma(f_i) \right\}$$

Recently a full characterisation of the connectedness of generalised inverse limits on intervals was given by Greenwood and Kennedy [2]. The proof relies on a notion of left/right sets and top/bottom sets which can not be easily translated to more general topological spaces. Work by Greenwood and Lockyer [3] takes inspiration from these ideas to construct a more general left and top set from the cartesian product of sets on Hausdorff continua. Giving sufficient conditions for the inverse limit to be disconnected. In this talk we investigate necessary conditions for disconnectedness. In particular we present a theorem that shows a disconnected inverse limit guarantees the existence of what we call generalised L and T sets.

References

[1] *An Introduction to Inverse Limits with Set-valued Functions*: W.T. Ingram, 2012.
[2] *Connectedness and inverse limits with set-valued functions on intervals*: Sina Greenwood, Judy Kennedy, 2013.
[3] *Connected generalised inverse limits over Hausdorff continua* : Sina Greenwood, Michael Lockyer, 2012.

Connections between saddle periodic orbits as organising centres of complicated dynamics

Nelson Wong
Supervisors: Bernd Krauskopf and Hinke M. Osinga

Dynamical systems theory investigates the behaviour of systems that change over time. Predictable behaviour such as steady states or periodic motion is well understood, but more complicated, higher-dimensional dynamics remains an active area of research. In particular, a four-dimensional system can have the following configuration:

1) A saddle periodic orbit with a three-dimensional invariant manifold, known as the "stable manifold", that comprises all trajectories converging to the periodic orbit in forwards time.

2) A saddle periodic orbit with a two-dimensional stable manifold.

3) A cycle of connecting orbits from one periodic orbit to the other, known as a "heterodimensional cycle".

Recent theory states that a dynamical system exhibiting a heterodimensional cycle can be perturbed to generate a robust cycle [1], which guarantees the existence of higher-dimensional chaos. However, a robust cycle is a complicated structure that replaces one of the saddle periodic orbits with a highly non-trivial saddle set, known as a "blender". Under certain geometric conditions, the blender contains the replaced periodic orbit after perturbation [2].

Since heterodimensional cycles are typically studied abstractly, there are very few known examples arising out of applications. One system that is known to feature a heterodimensional cycle is a four-dimensional model of intracellular calcium oscillations [3]. In this presentation, we show cycles between saddle periodic orbits that have been computed for the calcium model. We also discuss how one might computationally examine the geometry of a cycle and verify that a blender containing a perturbed periodic orbit exists.

References:

[1] Bonatti C., Díaz L. (2008) Robust heterodimensional cycles and C^1-generic dynamics. *Journal of the Institute of Mathematics of Jussieu* **7**(3):469-525.

[2] Bonatti C., Díaz L., Kiriki S. (2012) Stabilization of heterodimensional cycles. *Nonlinearity* **25**(4):931-960.

[3] Zhang W., Krauskopf B., Kirk V. (2012) How to find a codimension-one heteroclinic cycle between two periodic orbits. *Discrete Continuous Dynamical Systems - Series A* **32**(8):2825-2851.