Auckland Mathematical Olympiad 2015

Division 1

Questions

1. Is it possible to replace stars with plusses or minusses in the following expression

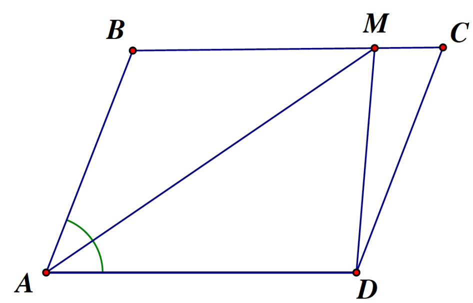
$$1 \star 2 \star 3 \star 4 \star 5 \star 6 \star 7 \star 8 \star 9 \star 10 = 0$$

so that to obtain a true equality.

2. On the table there are 2016 coins. Two players play the following game making alternating moves. In one move it is allowed to take 1, 2 or 3 coins. The player who takes the last coin wins. Which player has a winning strategy?

3. Several pounamu stones weigh altogether 10 tons and none of them weigh more than 1 tonne. A truck can carry a load which weight is at most 3 tons. What is the smallest number of trucks such that bringing all stones from the quarry will be guaranteed?

4. The bisector of angle A in parallelogram $ABCD$ intersects side BC at M and the bisector of $\angle AMC$ passes through point D. Find angles of the parallelogram if it is known that $\angle MDC = 45°$.



Auckland Mathematical Olympiad 2015

Division 2

Questions

5. The teacher wrote on the blackboard quadratic polynomial $x^2 + 10x + 20$. Then in turn each student in the class either increased or decreased by 1 either the coefficient of x or the constant term. At the end the quadratic polynomial became $x^2+20x+10$. Is it true that at certain moment a quadratic polynomial with integer roots was on the board?

6. A convex quadrillateral $ABCD$ is given and the intersection point of the diagonals is denoted by O. Given that the perimeters of the triangles ABO, BCO, CDO, ADO are equal, prove that $ABCD$ is a rhombus.

7. In the calculation

$$HE \times EH = WHEW,$$

where different letters stand for different nonzero digits. Find the values of all the letters.

8. In the planetary system of the star Zoolander there are 2015 planets. On each planet an astronomer lives who observes the closest planet into his telescope (the distances between planets are all different). Prove that there is a planet who is observed by nobody.

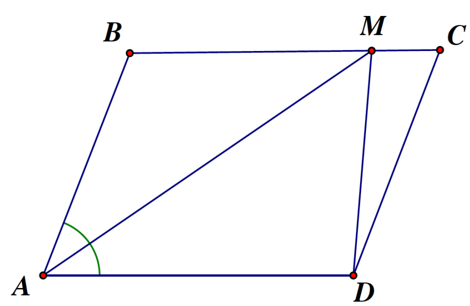
Solutions

1. If we replace all stars with plusses, we will get $1+2+3+4+5+6+7+8+9+10 = 55$. If we now change any plus with a minus, then the sum 55 will decrease by an even amount. The same will be true for all subsequent changes. Hence the result of all operations will be always odd and cannot be 0.

2. The second one. If the first player takes x coins, the second should take $4-x$. Then after a move of the first player the number of coins will never be divisible by 4, while it will be after every move of the second player. Hence it is the second player who takes the last coin.

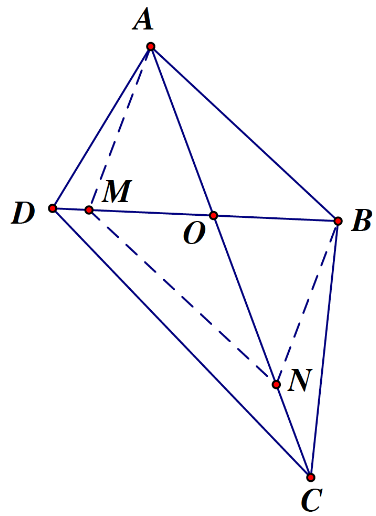
3. Answer: 5 trucks. Indeed, 5 trucks will be always enough. First four trucks can carry at least 8 tons of stones and the fifth will be able to carry all the rest. If there were 13 stones weighing 10/13 tons each, then each truck would be able to carry only 3 of them, hence five cars might be needed in this case.

4. Answer: $\angle A = 60^\circ$ and $\angle B = 120^\circ$. Let $\angle DAM = \angle BAM = \alpha$. Then, since $AD \parallel BC$, we have also $\angle AMB = \alpha$. Then $\angle AMC = 180^\circ - \alpha$ and hence $\angle CMD = 90^\circ - \frac{\alpha}{2}$. But then $\angle MDA = 90^\circ - \frac{\alpha}{2}$. Then $\angle MDC = \angle CDA - \angle MDA = (180^\circ - 2\alpha) - (90^\circ - \frac{\alpha}{2}) = 90^\circ - \frac{3}{2}\alpha$. Hence $90^\circ - \frac{3}{2}\alpha = 45^\circ$ and $\alpha = 30^\circ$.



5. At the beginning $f(-1) = 11$ and at the end $f(-1) = -9$. Each student increases or decreases this number by 1. Therefore at some stage $f(-1) = 0$ and $f(x)$ has integer roots.

6. Suppose that $AO \leq OC$ and $BO \leq OD$. Let M, N be the points on OC, OD such that $AO = ON$, $BO = OM$.



Then $ABMN$ is a parallelogram and the perimeters of ABO and MNO are equal. Therefore the perimeters of CDO and MNO are equal. It can happen only if $M = C$ and $N = D$, that is in the case when $ABCD$ is a parallelogram.

Now the perimeters of ABO and BCO are equal, whence $AB = BC$ and $ABCD$ is rhombus.

7. We have

$$EH \times HE = WHEW = W \times 1001 + 10 \times HE.$$

Hence

$$HE \times (EH - 10) = W \times 1001 = W \times 7 \times 11 \times 13.$$

Since $E \neq H$, HE is not divisible by 11, hence $EH - 10$ is divisible by 11, whence $E - 1 = H$. Now HE is a multiple of 13 with consecutive digits, hence it is 78. Then $W = 6$ and the unique reconstruction is

$$78 \times 87 = 6786.$$

8. Let us consider the two closest planets A and B. The astronomers on those observe each other planets. If anybody else observes A or B, then definitely there will be an unobserved planet. If nobody else observes A and B we will exclude them and will be left with 2013 planet system for which there will be 2013 astronomers observing these 2013 planets. Then we can identify another pair of planets whose astronomers look at each other planets and so on. Eventually we will be left with just one planet which is observed by nobody.