2018 AUCKLAND MATHEMATICAL OLYMPIAD

Questions

- Write all your working and solutions below the question.
- You are expected to show how you obtained your solution for each question, as correct answers (without working) will be awarded 1 mark only.
- Markers will mark only one question at a time per candidate.

Junior Division

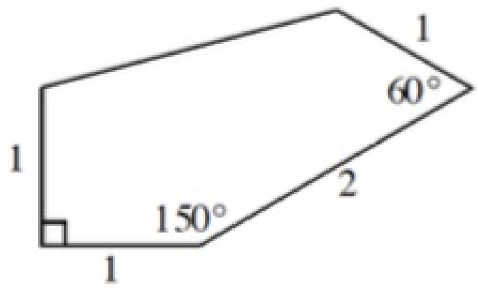
1. Find a multiple of 2018 whose decimal expansion's *first* four digits are 2017.

2. Starting with a list of three numbers, the "*Make-My-Day*" procedure creates a new list by replacing each number by the sum of the other two.

For example, from $\{1, 3, 8\}$ "*Make-My-Day*" gives $\{11, 9, 4\}$ and a new "*Make-My-Day*" leads to $\{13, 15, 20\}$.

If we begin with $\{20, 1, 8\}$, what is the maximum difference between two numbers on the list after 2018 consecutive "*Make-My-Day*"s?

3. Consider the pentagon below. Find its area.



4. A vintage tram departs a stop with a certain number of boys and girls on board.
At the first stop, a third of the girls get out and their places are taken by boys.
At the next stop, a third of the boys get out and their places are taken by girls.
There are now two more girls than boys and as many boys as there originally were girls.
How many boys and girls were there on board at the start?

5. Find all possible triples of positive integers, a, b, c so that $\frac{a+1}{b}$, $\frac{b+1}{c}$, and $\frac{c+1}{a}$ are also integers.

Senior Division

6. For two non-zero real numbers a, b, the equation, $a(x-a)^2 + b(x-b)^2 = 0$ has a unique solution.

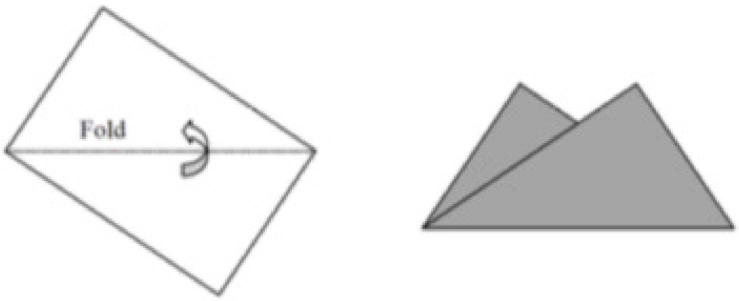
Prove that $a = \pm b$.

7. Consider a positive integer,

$$N = 9 + 99 + 999 + \ldots\ldots + \underbrace{999\ldots9}_{2018}$$

How many times does the digit 1 occur in its decimal representation?

8. A rectangular sheet of paper whose dimensions are 12 x 18 is folded along a diagonal, creating the M-shaped region drawn in the picture (see below).



Find the area of the shaded region.

9. Alice and Bob are playing the following game:

They take turns writing on the board natural numbers not exceeding 2018 (to write the number twice is forbidden).

Alice begins. A player wins if after his or her move there appear three numbers on the board which are in arithmetic progression.

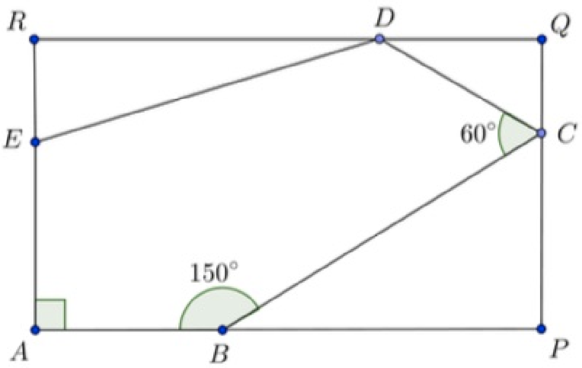
Which player has a winning strategy?

10. There is a sequence of numbers +1 and -1 of length n. It is known that the sum of every 10 neighbouring numbers in the sequence is 0 and that the sum of every 12 neighbouring numbers in the sequence is not zero.

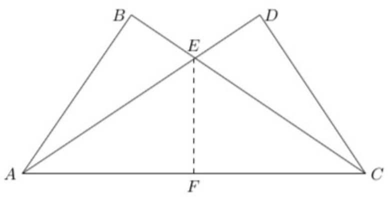
What is the maximal value of n?

2018 AUCKLAND MATHEMATICAL OLYMPIAD

Solutions for Students

1	$2018 \times 9996 = 20171928$
2	Claim: 19. In fact, the "*MakeMyDay*" procedure does not change the maximum difference between two numbers on the list. Suppose our list is $\{a,b,c\}$ with $a < b < c$. The maximum difference between the largest and the smallest number is $c - a$. The "*MakeMyDay*" operation creates $\{b + c, a + c, a + b\}$. Since $a < b$, we know that $a+c < b+c$. Since $b < c$, we also know that $a+b < a+c$. Combining these two inequalities, we have $a + b < a + c < b + c$. The maximum difference between any number is $(b+c) - (a+b)$ or $c - a$. So the same as the one we started with. For the initial list of {20,1,8}, the maximum difference will always be 19.
3	Inscribe in a rectangle, where ∠CBP=30deg.  Since $\overline{BC}$ = 2, we get $\overline{CP}$ = 1, and $\overline{BP} = \sqrt{3}$, So $\overline{AP} = 1 + \sqrt{3}$. Then ∠DCQ = 60deg, so $\overline{CQ}$ = 1/2, $\overline{PQ}$=3/2, $\overline{DQ} = (\sqrt{3})/2$. Finally, $\overline{DR} = 1 + \sqrt{3} - (\sqrt{3})/2 = \frac{2+2\sqrt{3}-\sqrt{3}}{2} = 1 + (\sqrt{3})/2$, and $\overline{ER}$ = 1/2. The area of the pentagon is then $\overline{AP} \text{ x } \overline{PQ} - 1/2(\overline{BP}\text{x}\overline{CP} + \overline{CQ}\text{x}\overline{DQ} + \overline{DR}\text{x}\overline{ER})$ $= 1/4(5+3\sqrt{3})$ = approx 2.549 units^2
4	Let b_i and g_i be the numbers of boys and girls on board the tram, respectively, at stop i. Note that b_0 and g_0 are the numbers of boys and girls on board the tram, respectively, at the start of the trip. At stop 1, b_1=b_0 + g_0/3, g_1=2 g_0/3. Similarly, at stop 2, b_2 = 2 b_1/3, g_2 = g_1+b_1/3. Using above formulas for b_1, g_1 leads to b_2 = 2b_0/3 + 2g_0/9, g_2 = 7g_0/9 + b_0/3. From b_2 = g_0 this yields b_0 = 7g_0/6. As b_2+2 = g_2, we have g_0 = 1/5(3b_0+18). We can now solve for g_0 to get g_0=12 and then b_0=14.
5	We have $b \le a+1$, $c \le b+1$, $a \le c+1$, so that $c-1 \le b \le a+1 \le c+2$, so a in $\{c-2, c-1, c, c+1\}$. Now a case bash yields 10 solutions: (1,2,3),(3,1,2),(2,3,1), (3,4,5),(5,3,4),(4,5,3), (2,1,1),(1,2,1),(1,1,2), (1,1,1).

<table>
<tr><td>6</td><td>Suppose a and b are both positive. Then $a(x-a)^2 \geq 0$ and $b(x-b)^2 \geq 0$
with the first being zero for x = a and the second being zero for x = b.
Thus $a(x-a)^2 + b(x-b)^2 = 0$ only when x = a = b.
If a and b are both negative, then the reasoning is similar.
If a and b have opposite signs, we rewrite the equation as follows:
$$a(x-a)^2 + b(x-b)^2 = (a+b)x^2 - 2(a^2+b^2)x + (a^3+b^3) = 0$$
Then its discriminant, D $= 4(a^2+b^2)^2 - 4(a+b)(a^3+b^3)$
$= 4(2a^2b^2 - ab^3 - ba^3)$
$= -4ab(-2ab + b^2 + a^2)$
$= -4ab(a-b)^2 < 0$, is negative unless $b = -a$.</td></tr>
<tr><td>7</td><td>We have
N $= (10-1) + (10^2-1) + \cdots + (10^{2018}-1) = \underbrace{111\ldots10}_{2018} - 2018$
$= \underbrace{111\ldots100000}_{2014} + (11110 - 2018)$
$= \underbrace{111\ldots100000}_{2014} + 9092.$
Hence in decimal representation there are 2014 ones.</td></tr>
</table>

<table>
<tr><td>8</td><td>The area of the M-region is area of rectangle ABCD - area ΔACE as triangle ΔACE is the overlap.


|ABCD| = 12 x 18 = 216.
$\overline{AC}$ is the diagonal of the rectangle,
so $\overline{AC} = \sqrt{12^2 + 18^2}$
$= 6\sqrt{13}$.
The height of ΔACE is $\overline{EF}$,
and equals $\overline{FC}$ x tan $\angle DAC$
$|\overline{FC}| = 3\sqrt{13}$
and tan $\angle DAC = \frac{12}{18} = \frac{2}{3}$
So $|\Delta ACE| = \frac{1}{2}|\overline{AC}|$ x $|\overline{EF}|$ = 78.
Area = 216-78 = 138.</td><td>OR . . .
With ΔADC = half rectangle $ABCD = 108$
then remaining shaded area = ΔABE,
and since side $\overline{AB}$ = 12,
we can find $\overline{BE}$ from 12 × tan($\angle BAE$),
where $\angle BAE$ is from $\angle BAC - \angle DAC$
(i.e. from $tan^{-1}\left(\frac{18}{12}\right) - tan^{-1}\left(\frac{12}{18}\right)$)

So $\overline{BE}$ = 5,
and smaller ΔABE = 30, so combined shaded area of $\Delta ADC + \Delta ABE$
= 108 + 30 = 138.</td><td>OR . . .
We have
$\overline{AE}$ (= $\overline{EC}$) + $\overline{ED}$ = 18,
so set up the equation with $\overline{ED} = \overline{EB} = x$
so that
$x^2 + 12^2 = (18-x)^2$
$144 = 324 - 36x$
Solves $x = 5 = \overline{ED}$.
Therefore the shaded area
$= \frac{1}{2}$(12 x 18) + $\frac{1}{2}$(5 x 12)
= 138.</td></tr>
</table>

9	If after her second move Alice does not win, then Bob wins with his second move. Indeed, in this case there are two numbers *a* and b on the board with the same parity. Bob wins with writing $\frac{1}{2}(a+b)$. We just need to show how Bob chooses his first move. If Alice chooses $a \leq 1009$, then Bob chooses the number in the set $\{2017, 2018\}$ whose parity is different from *a*. And, if Alice chooses $a \geq 1010$, then Bob chooses *b* = 1 or 2, the one whose parity is different from *a*. Then Alice cannot win with her second move since $\frac{1}{2}(a+b)$ is not an integer.
10	Firstly, let us show that *n* = 15 is possible. Indeed, we can have the sequence +1 + 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1 + 1 + 1 + 1 + 1 + 1: Let us prove that *n* cannot be larger. Suppose $n \geq 16$ and our sequence is $x = x_1x_2x_3 \dots x_n$. Without loss of generality suppose $x_1 = 1$. Then, as the sum of every 10 neighbouring numbers is 0, we have $x_{11} = 1$. Thus we have $x = 1x_2x_3 \dots x_{10}1x_{12}x_{13} \dots x_n$. We claim that $x_{12} = \dots = x_n = 1$. Indeed, among these, -1 cannot follow 1 since then we will have the sum of 12 consecutive terms of the sequence, which ends with these +1 and -1 being 0. If $n \geq 16$, then we have at least 6 ones at the end of the sequence which it then makes impossible to have the sum of the last 10 terms to be 0.