Auckland Mathematical Olympiad 2019

Junior Division

Questions and Solutions

1. Given a convex quadrilateral $ABCD$ in which $\angle BAC = 20°$, $\angle CAD = 60°$, $\angle ADB = 50°$, and $\angle BDC = 10°$. Find $\angle ACB$.

Solution. The triangle $\triangle ACD$ is equilateral since $\angle CAD = 60°$ and also $\angle CDA = \angle CDB + \angle ADB = 50° + 10° = 60°$. Hence $AC = CD = AD$. Now the triangle $\triangle BAD$ is isosceles since $\angle ADB = \angle ABD = 50°$ and hence $AB = AD = AC$ from which we imply that $\triangle ABC$ is also isosceles. Thus

$$\angle ABC = \angle ACB = \frac{180° - 20°}{2} = 80°. \qquad \square$$

2. There are 2019 segments $[a_1, b_1], \ldots, [a_{2019}, b_{2019}]$ on the line. It is known that any two of them intersect. Prove that they all have a point in common.

Solution. Let $n = 2019$. Let a_i is maximal among $a_1, \ldots, a_n$ and b_j is the smallest among $b_1, \ldots, b_n$. Then we must have $b_j \geq a_i$. Hence the segment $[a_i, b_j]$ belongs to all the intervals. $\square$

3. Let x be the smallest positive integer that cannot be expressed in the form $\dfrac{2^a - 2^b}{2^c - 2^d}$, where a, b, c, d are non-negative integers. Prove that x is odd.

Solution. Let $x = 2x'$, where x' is an integer. Then x' can be represented in the form $x' = \dfrac{2^a - 2^b}{2^c - 2^d}$. But then $x = \dfrac{2^{a+1} - 2^{b+1}}{2^c - 2^d}$ which contradicts to minimality of x. $\square$

4. Suppose $a_1 = \frac{1}{6}$ and

$$a_n = a_{n-1} - \frac{1}{n} + \frac{2}{n+1} - \frac{1}{n+2}$$

for $n > 1$. Find a_{100}.

Solution. We have

$$a_2 = a_1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4}$$

and

$$a_3 = a_2 - \frac{1}{3} + \frac{2}{4} - \frac{1}{5} = a_1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$$

Further

$$a_4 = a_1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{6}$$

The pattern is now clear and (ideally we can prove it by by induction)

$$a_k = a_1 - \frac{1}{3} + \frac{1}{4} + \frac{1}{k+1} - \frac{1}{k+2}.$$

In particular,

$$a_{100} = \frac{1}{6} - \frac{1}{3} + \frac{1}{4} + \frac{1}{101} - \frac{1}{102} = \frac{1}{10302}.$$

□

5. 2019 coins are on the table. Two students play the following game making alternating moves. The first player can in one move take the odd number of coins from 1 to 99, the second player in one move can take an even number of coins from 2 to 100. The player who can not make a move is lost. Who has the winning strategy in this game?

Proof. We describe the winning strategy of the first player. The first move is to take 99 coins from the table. Each next move, if the second player takes x coins, then the first player must takes $101 - x$ coins. (he can always do this, because if x is an even number from 2 to 100, then $(101 - x)$ is an odd number from 1 to 99). Since 2019 = 101 x 19 + 99 + 1, after 19 such moves, after the move of the first player, only 1 coin will remain on the table, and the second player will not be able to make a move, hence lose. □

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Senior Division

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6. Function f satisfies the equation $f(\cos x) = \cos(17x)$. Prove that it also satisfies the equation $f(\sin x) = \sin(17x)$.

Solution. We have

$$f(\sin x) = f(\cos(\frac{\pi}{2} - x) = \cos(\frac{17\pi}{2} - 17x) = \cos(\frac{\pi}{2} - 17x) = \sin(17x). \qquad \square$$

7. Prove that among any 43 positive integers there exist two a and b such that $a^2 - b^2$ is divisible by 100.

Solution. We observe that x^2 and $(100 - x)^2 = 100(100 + 2x) + x^2$ give the same remainder on division by 100. Also, the number x and $100k + x$ have the same remainder on division by 100. Hence a square may have only 51 possible remainder on division by 100.

If among these remainders we have three multiples of 5, then the difference of squares of their respective numbers is divisible by 25. Suppose they are $a = 5k$, $b = 5\ell$ and $c = 5m$. Then two of the numbers k, ℓ, m will have the same parity, say k and ℓ do. Then $a^2 - b^2 = (a - b)(a + b)$ will be divisible by 25 and 4, hence by 100. We have 11 remainders which are divisible by 5, hence 9 of them cannot be used. Thus we have $51 - 9 = 42$ remainders available but 43 numbers. $\square$

8. There is a finite number of polygons in a plane and each two of them have a point in common. Prove that there exists a line which crosses every polygon.

Solution. Project all polygons onto a certain line ℓ. The projection of any polygon is a segment and each two of these segments intersect. By solution of Q3 from the junior division all of them have a common point. Then the line perpendicular to ℓ drawn through the common point of all projections will intersect all polygons. $\square$

9. Find the smallest positive integer that cannot be expressed in the form $\dfrac{2^a - 2^b}{2^c - 2^d}$, where a, b, c, d are non-negative integers.

Solution. The numbers $1, 3, 5, 7, 9$ can be represented. For example, $5 = \frac{2^4-1}{2^1-1}$ and $9 = \frac{2^6-1}{2^3-1}$. But 11 cannot. Indeed, Note that the 4 parameters can be simplified into three:

$$\frac{2^a - 2^b}{2^c - 2^d} = 2^{b-d}\frac{2^{a-b} - 1}{2^{d-c} - 1} = 2^m \frac{2^r - 1}{2^s - 1}.$$

Since 11 is odd, $m = 0$. If we had $11 = \frac{2^r-1}{2^s-1}$, then $11(2^s - 1) = 2^r - 1$ which is not possible since all three numbers 11, $2^s - 1$ and $2^r - 1$ have remainder 3 on division by 4. $\square$

10. 2019 circles split a plane into a number of parts whose boundaries are arcs of those circles. How many colors are needed to color this geographic map if any two neighboring parts must be coloured with different colours?

Solution. Each point lies either within an even number of circles (area of type 1), or inside an odd number of them (area of type 2). Crossing an arc we move from area of one type to an area of another type. Therefore, two colors would be enough colouring areas of one type, say with a red colour and areas of another type with a blue colour. $\square$
