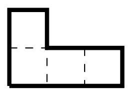
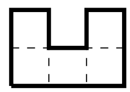
Auckland Mathematical Olympiad

Problems and Solutions

1. Find the smallest 4-digit number N such that $3N$ has only even digits.

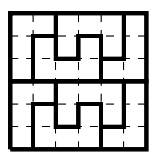
   Solution. Let $x \geq 1000$ be this number. Then $3x \geq 3000$. Moreover, $3x \geq 4000$ because $3x$ has only even digits. The smallest number above 4000 divisible by 3 is 4002, so $x = \frac{4002}{3} = 1334$. □

2. There are two types of tiles

    and .

   consisting of four and five unit squares, respectively. Heather used equal number of these tiles to cover a square $n \times n$ grid also made of unit squares. What is the smallest n for which she could do this?

   Solution. Let k be the number of tiles of each type used for covering the $n \times n$ grid. Together, they cover $(4 + 5)k = 9k$ unit squares. Thus, $n^2 = 9k$, so n is divisible by 3. Clearly, one cannot cover the 3×3 grid with one tile of each type. On the other hand, one can cover the 6×6 grid as follows:

   □

3. There are 7 lines on a plane, no two of them are parallel. Prove that there are two lines such that the angle between them is less than $26°$.

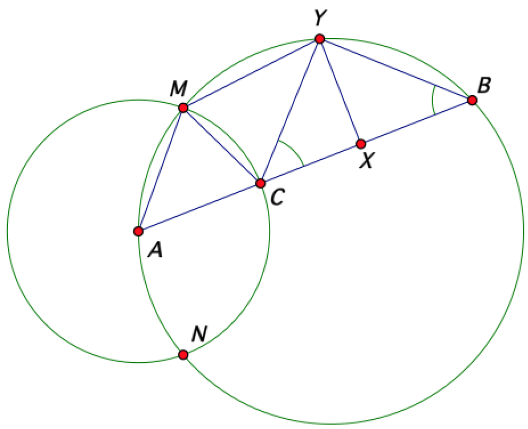
   Solution. If a line moves remaining parallel to itself, all angles remain as they were. Thus we can move our lines so that they all pass through the same fixed point. Then they form 14 angles. Since $26° \times 14 = 364° > 360°$, one angle must be smaller that $26°$. □

4. In the vicinity of the wharf, the river has several islands with total perimeter 8 km. The banks of the river are parallel and the width of the river is 1 km. Is it true that starting from the wharf one can sail to the other bank covering a distance less than 3 km?

   Solution. Yes. The total length of projections of islands onto the bank with the wharf is strictly less than 4 km. Therefore the closest clear space will be within 2 km from the wharf to the left or the right. Hence, we first go to the place not covered by the projection, and then cross the river orthogonally to the bank, covering another 1 km. □

5. A circle γ with a chord AB is given. Another circle δ centred at A intersects γ at points M, N and the chord AB at C. Prove that the perpendicular bisector of the segment BC intersects the arc MB, not containing A, at the midpoint of that arc.

Solution. Let X be the midpoint of BC and Y be the point of intersection of the arc MB, not containing A, with the perpendicular bisector of BC.



Then, $|YB| = |YC|$ and $\angle YBC = \angle YCB$. Since B lies on the arc AY, not containing M, we have $\angle AMY + \angle YBC = 180°$ and thus $\angle AMY = \angle ACY$. But $|AM| = |AC|$, and thus $\angle AMC = \angle ACM$. It follows that $\angle YMC = \angle YCM$ and $|YC| = |YM|$. Therefore $|YB| = |YC| = |YM|$ and Y is the midpoint of the arc MB. $\square$

6. Prove that for all $x \geq 0$ the following inequality holds

$$2^{\sqrt[12]{x}} + 2^{\sqrt[4]{x}} \geq 2 \cdot 2^{\sqrt[6]{x}}.$$

Solution. We apply AM-GM inequality two times

$$2^{\sqrt[12]{x}} + 2^{\sqrt[4]{x}} \geq 2 \cdot 2^{\frac{1}{2}\left(\sqrt[4]{x} + \sqrt[12]{x}\right)} \geq 2 \cdot 2^{\sqrt[6]{x}}. \qquad \square$$

7. 64 real numbers are written in the cells of the 8×8 table. It is allowed to simultaneously change the signs of all numbers in one column or in one row. Prove that in several such operations it is possible to achieve that the sums of the numbers in every row and every column are non-negative.

Solution. Let us organise the following process: if the sum of the numbers in some row (or in some column) is negative, then we change the signs of all the numbers in this row (this column). After each such step, the sum of all the numbers in the table increases. Since the number of possible arrangements of signs of numbers in the table is finite, the process ends after finitely many steps, and thus leads to the sums of the numbers in any row and any column becoming non-negative, which is what was required to be proved. $\square$

8. Prove that $(((3!)!)!$ is the number with more than 1000 decimal digits.

Solution. We have $3! = 6$, $(3!)! = 6! = 720$, $(((3!)!)! = 720!$. Since

$$720! > 100 \cdot 101 \cdot \ldots \cdot 720 > 100^{621} > 10^{1242}$$

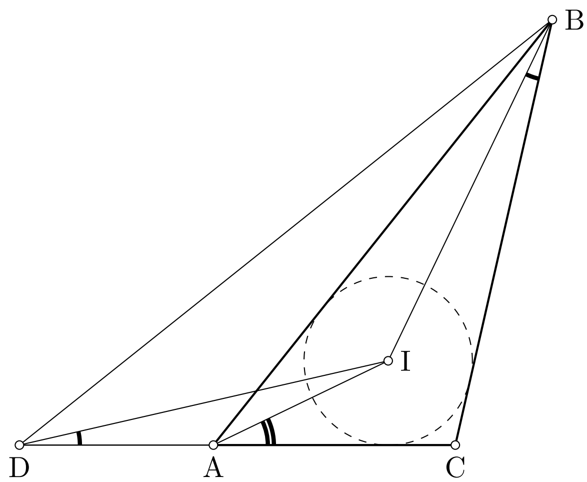
which means $(((3!)!)!$ has at least 1242 digits. $\square$

9. Find all functions $f\colon [0,\infty) \to [0,\infty)$ such that $(y+1)f(x+y) = f(xf(y))$.

   Solution. Inserting $x = 0$ yields $f(y) = \frac{f(0)}{y+1}$. Plugging this formula back in the original equation gives
   $$(y+1)\frac{f(0)}{x+y+1} = \frac{f(0)}{x\frac{f(0)}{y+1}+1}$$
   for all $x, y \geq 0$, which simplifies to $\frac{f(0)}{x+y+1} = \frac{f(0)}{f(0)x+y+1}$. This holds for all $x, y \geq 0$ precisely when $f(0) = 0$ or $f(0) = 1$. Thus, $f = 0$ and $f = \frac{1}{x+1}$ are the sought functions. $\square$

10. Let I be the centre of the inscribed circle of a triangle ABC, and suppose $|AC| + |AI| = |BC|$. What is the ratio between angles $\angle ABC$ and $\angle BAC$?

    Solution. Since I is the centre of the inscribed circle, we have $\angle ABC = 2\angle IBC$ and $\angle BAC = 2\angle IAC$. Let us extend the side AC beyond A to the point D such that $|AD| = |IA|$. Then the triangles DCB and DAI are isosceles. From the first one we see $\angle IBC = \angle IDC$; from the second one we see $\angle IAC = 180° - \angle IAD = 2\angle IDC$. Hence, the ratio is $1 : 2$.



$\square$

11. Prove that the equation
    $$\sin\cos x = \cos\sin x$$
    does not have real solutions.

Solution. This equation can be written as

$$\sin\cos x = \sin\left(\frac{\pi}{2} - \sin x\right)$$

If $\sin a = \sin b$, then either $a - b = 2\pi k$ or $a + b = (2k+1)\pi$. Hence either

$$\sin x + \cos x = \frac{\pi}{2} + 2k\pi$$

or

$$\sin x - \cos x = \frac{\pi}{2} + (2k+1)\pi.$$

The first can be written

$$\sin\left(\frac{\pi}{4} + x\right) = \pi\left(\frac{1}{2\sqrt{2}} + 2k\right).$$

This can never be satisfied because the right-hand-side is never smaller than 1 in absolute value (since $3 \cdot \frac{1}{2\sqrt{2}} > 1$). A similar conclusion holds for the second equation. □

12. Let n be a positive integer. The function f is defined as follows. If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

then

$$f(n) = 1 + \alpha_1 p_1 + \ldots + \alpha_r p_r.$$

Prove that for any $n > 6$ in the sequence n, $f(n)$, $f(f(n))$, $\ldots$ eventually will appear number 8.

Solution. If $n \geq 9$ is composite, then $f(n) \leq n - 2$. Indeed, suppose $n = n_1 n_2$ with prime decompositions

$$n_1 = p_1^{\alpha_1} \ldots p_r^{\alpha_r}, \qquad n_2 = p_1^{\beta_1} \ldots p_r^{\beta_r}$$

(we may consider that the same primes are involved allowing exponents to be 0). Then

$$f(n) = f(n_1 n_2) = 1 + (\alpha_1 + \beta_1)p_1 + \ldots + (\alpha_r + \beta_r)p_r = f(n_1) + f(n_2) - 1.$$

Hence

$$f(n) = f(n_1)+f(n_2)-1 \leq (n_1+1)+(n_2+1)-1 = n_1+n_2+1 = (n+2)-n_1n_2+n_1+n_2-1 =$$

$$(n+2) - (n_1 - 1)(n_2 - 1).$$

Since $n = n_1 n_2 \geq 9$, then $(n_1 - 1)(n_2 - 1) \geq 4$, hence

$$f(n) = n + 2 - (n_1 - 1)(n_2 - 1) \leq n + 2 - 4 = n - 2.$$

If n is prime, then $n + 1$ is composite and hence $f(f(n)) = f(n+1) \leq n - 1$.

Thus eventually in the sequence n, $f(n)$, $f(f(n))$, $\ldots$ we will find the number smaller than 9. In this sequence we cannot find the number smaller than 7 since if $f(n) < 7$, then $n < 7$. Hence we will eventually obtain one of the numbers 7 or 8. But $f(7) = 8$ and we are done. □