BEYOND CHAOS:
Is there a wild butterfly effect?

Juan Patiño-Echeverría

Department of Mathematics, University of Auckland

Introduction

Have you heard about the butterfly effect? The idea that the flap of a butterfly's wings in Brazil could set off a tornado in Texas?

This concept was discovered by Lorenz in the 1960s [2]. He illustrated it with a simple model of convection dynamics in the atmosphere, given by the vector field:

$$\begin{cases} \dot{x} = \sigma(y-x), \\ \dot{y} = \rho x - xz - y, \\ \dot{z} = xy - \beta z, \end{cases} \tag{1}$$

for the parameters $\rho = 28$, $\beta = 8/3$ and $\sigma = 10$. When starting from two points arbitrarily close together, model (1) produces a flow that pushes these points far apart very quickly, even though both solutions lie on a butterfly-shaped *strange attractor* (Figure 1), which is arguably the most famous example of a (classical) chaotic attractor.

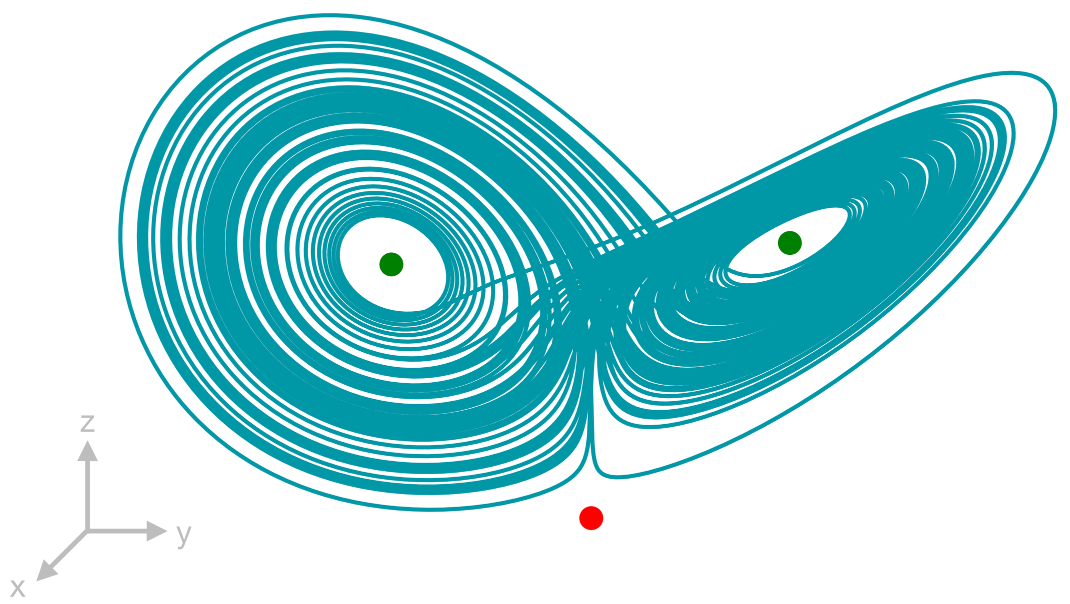

Figure 1. Lorenz attractor for $\rho = 28$, $\beta = 8/3$ and $\sigma = 10$. The red point is the equilibrium at the origin, and the green points are a symmetric pair of equilibria.

Our research is focused on characterising and identifying possible parameter-dependent transitions to ***wild chaos,*** a new type of chaotic dynamics that can only arise in flows of dimension at least four.

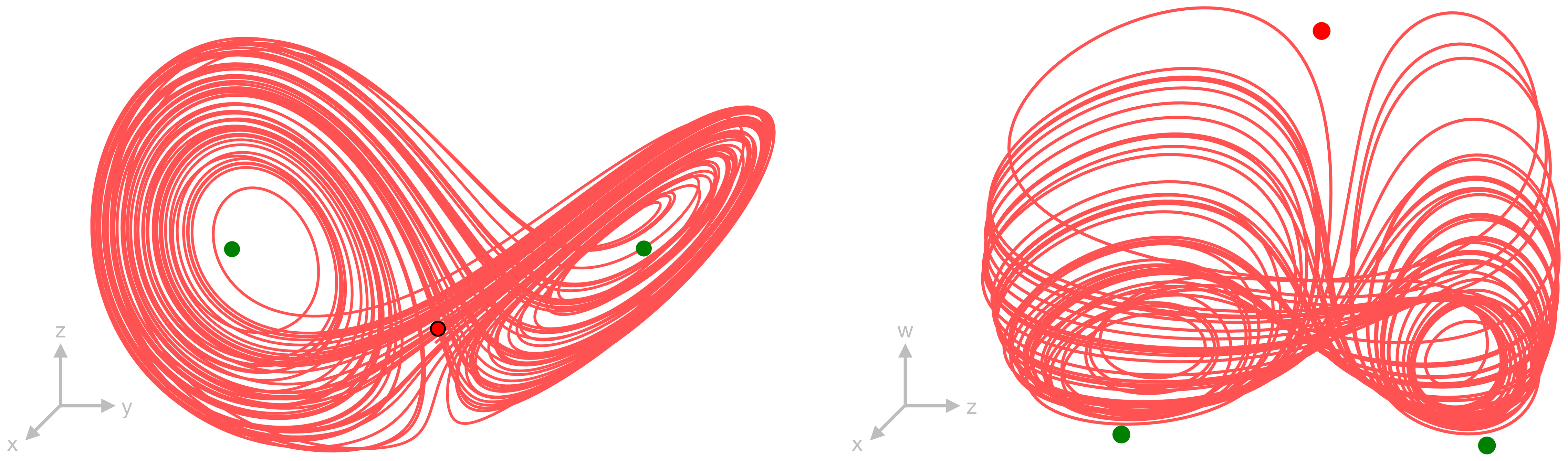

Figure 2. Projections of the wild chaotic attractor existing in model (2) at $\rho = 25$, $\beta = 8/3$, $\sigma = 10$ and $\mu = 7$ onto the (x, y, z)-plane (left) and the (x, z, w)-plane (right). The red points represent the equilibrium at the origin, and the green points a symmetric pair of equilibria.

Overview

We study the four-dimensional Lorenz-like model:

$$\begin{cases} \dot{x} = \sigma(y-x), \\ \dot{y} = \rho x - xz - y, \\ \dot{z} = xy - \beta z + \mu w, \\ \dot{w} = -\beta w - \mu z, \end{cases} \tag{2}$$

which has a *wild chaotic attractor* according to [1] for the values $\rho = 25$, $\beta = 8/3$, $\sigma = 10$ and $\mu = 7$ (Figure 2).

- **Why these parameter values?**
- **Why is this wild chaotic attractor different from a classical chaotic attractor?**
- **What invariant sets are involved to create this wild chaos?**

We want to answer these questions using a geometric approach to study changes in the topology of model (2) as ρ and μ vary.

Methods

We numerically catalogue changes to the number and stability of equilibrium and oscillating solutions in a *bifurcation diagram* (Figure 3). Each time a curve in the (ρ,μ)-plane is crossed, model (2) undergoes a topological change in its dynamics.

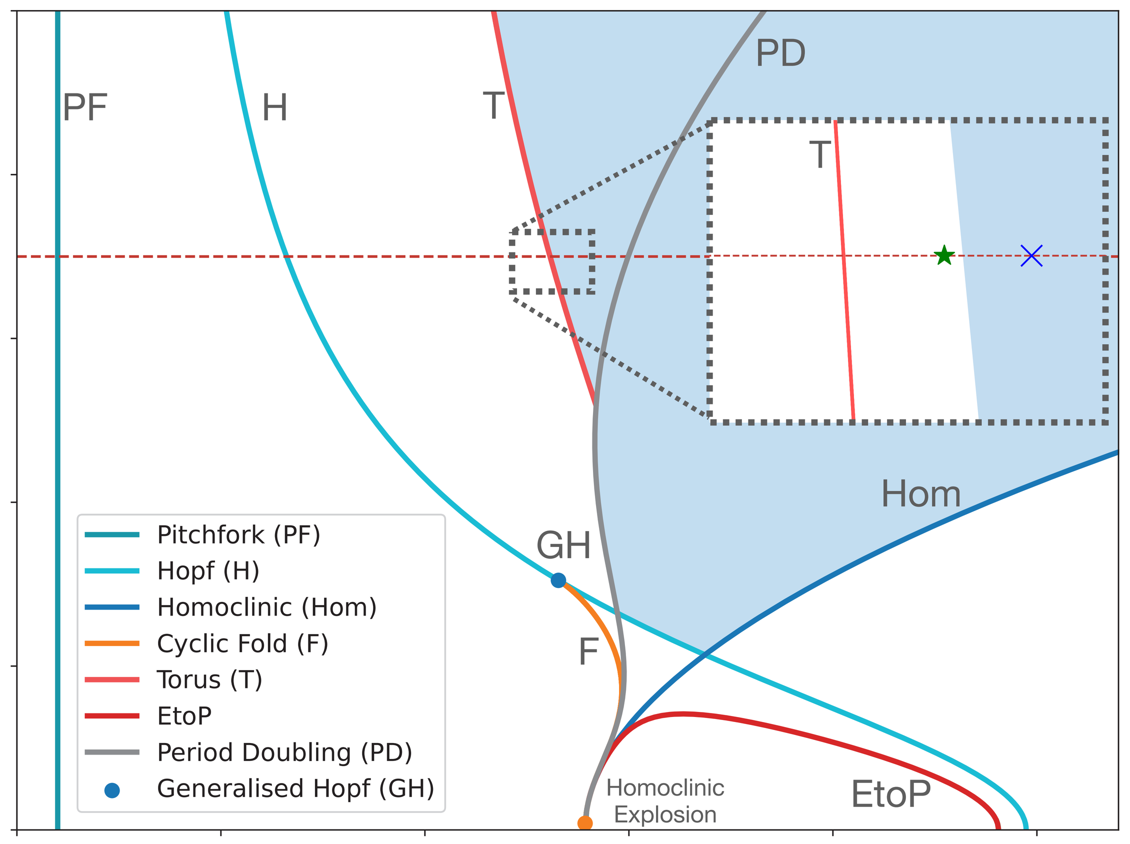

Figure 3. Bifurcation Diagram. The (ρ, μ)-plane is divided into regions bounded by bifurcation curves that signify a change in topology. The shaded region is a possible candidate for wild chaos.

Results and Future Work

The bifurcation diagram in Figure 3 is not yet complete. Inside the shaded region, there are qualitative changes in the dynamics of model (2).

For instance, consider the parameter points ★ and X in the enlargement in Figure 3. For ★, the attractor is a *quasi-periodic torus*, whereas, for X, it is *phase-locked* (Figure 4).

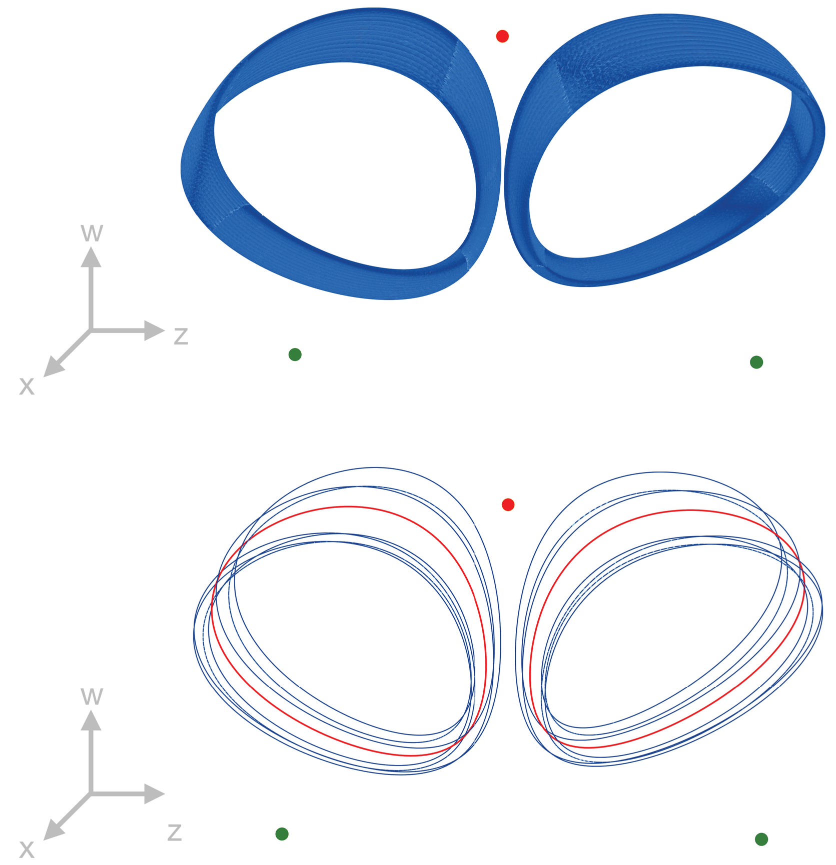

Figure 4. Projections of a quasi-periodic torus (top) for $\rho = 13.074$, and a phase-locked torus for $\rho = 13.08$ (bottom). In both cases $\beta = 8/3$, $\sigma = 10$ and $\mu = 7$. The red points represent the equilibrium at the origin, and the green points a symmetric pair of equilibria.

- Existence of an attracting invariant torus is not possible in the classical model (1).
- Contrary to our expectation, wild chaos appears to originate at $\mu > 0$ rather than at the homoclinic explosion at $\mu = 0$.

References

[1] Gonchenko S V, Kazakov A O, Turaev D 2021 Wild pseudohyperbolic attractor in a four-dimensional Lorenz system *Nonlinearity* **34** 2018-47.

[2] Lorenz E N 1963 Deterministic non periodic flows *J. Atmos. Sci.* **20** 130.

Supervisors: Professors Hinke Osinga and Bernd Krauskopf

Contact: j.patino@auckland.ac.nz